Scale Invariance in the Causal Approach to Renormalization Theory

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Abstract

The dilation invariance is studied in the framework of Epstein-Glaser approach to renormalization theory. Some analogues of the Callan-Symanzik equations are found and they are applied to the scalar field theory and to Yang-Mills models. We find the interesting result that, if all the fields of the theory have zero masses, then from purely cohomological consideration, one can obtain the anomalous terms of logarithmic type.

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1 Introduction

The causal approach to renormalization theory of by Epstein and Glaser [24], [25] leads to important simplification of the renormalization theory as well as of the computational aspects. This approach works for quantum electrodynamics [42], Yang-Mills theories [12] [13] [15] [16] [1] [2] [10], [11], [33]-[36], [37] [38] [23], gravitation [26], [27], [45], etc.

In this paper we investigate the rôle of dilation invariance in the causal approach. The pioneering works on scale covariance in perturbative field theory are [6], [7] and [8]. A mathematical refined analysis was developed in [43] and [44], the main mathematical tool being the so-called quantum action principle [39] (for a review see [41]).

Our strategy will be based exclusively on the Epstein-Glaser construction of the chonological producs for the free fields. In the next Section we define the dilation invariance operator for various free fields. Next, we remind the basic facts about renormalization theory. We will emphasize the original Epstein-Glaser approach where one considers a set of (linearly independent) interaction Lagrangian and attaches to each of this Lagrangian a (space-time dependent) coupling constant. Then we are able to prove the basic theorem concerning the arbitrariness of the chronological products for the same set of interaction Lagrangian. This problem was already addressed in [5], [40], but we argue that the natural framework is the multi-valued coupling constant approach of [24].

In Section 4 we obtain consequences about the scale behaviour of the chronological products. One expects that the action of the dilations operators on the chronological product should give the usual scaled chronological products, up to some *scale anomalies*. We will prove that one can reduce the analysis to a cohomological problem. An important difference appears in the study of this problem in the cases of a massless and of a massive field. If there exists at least one massive field in the theory then we can prove that one can choose the chronological product to be scale covariant. In the opposite case, one finds out that some anoumalous terms of logarithmic behaviour can appear. We emphasis again that these results hold for the chronological products of the free fields. We will comment on what one should expect for the case of the interacting fields in the last Section.

We will apply these considerations for Yang-Mills models in Section 5 and obtain a restrictions on the possible form of the anomalies, namely the canonical dimension of such an anomalous expression must be 5.

2 Dilation Invariance in Quantum Field Theory

It is well known that the Fock space of the real scalar field of mass m can be defined as:

$$\mathcal{F}_m \equiv \oplus_{n=0}^{\infty} \mathcal{F}_m^{(n)} \tag{2.0.1}$$

where $\mathcal{F}_m^{(n)}$ is the set of Borel function $\Phi^{(n)} : (X_m^+)^{\otimes n} \to \mathbb{C}$ which are square integrable with respect to the Lorentz invariant measure: $d\alpha_m^+(p) \equiv \frac{d\mathbf{p}}{\sqrt{\mathbf{p}^2+m^2}}$ and completely symmetric in the all variables (see [46] for notations). Then we have:

Proposition 2.1 *Let us define for any $\lambda \in \mathbb{R}_+$ the operators $\mathcal{U}_\lambda : \mathcal{F}_m \longrightarrow \mathcal{F}_{\lambda^{-1}m}$ as follows:*

$$\left(\mathcal{U}_\lambda \Phi^{(n)}\right)(p_1, \ldots, p_n) = \lambda^n \quad \Phi^{(n)}(\lambda p_1, \ldots, \lambda p_n). \tag{2.0.2}$$

Then:
(i) The operators $\mathcal{U}_\lambda$ are unitary;
(ii) The following relations are verified for all $\lambda, \lambda' \in \mathbb{R}_+$:

$$\mathcal{U}_\lambda \mathcal{U}_{\lambda'} = \mathcal{U}_{\lambda\lambda'}; \tag{2.0.3}$$

(iii) If $\mathcal{U}_{a,L}^{[m]}$ is the representation of the Poincaré group in the Fock space $\mathcal{F}_m^{(n)}$, then:

$$\mathcal{U}_{a,L}^{[m]} \quad \mathcal{U}_\lambda = \mathcal{U}_\lambda \quad \mathcal{U}_{\lambda^{-1}a,L}^{[\lambda m]} \tag{2.0.4}$$

for all translations a and all Lorentz transformations L.

Proof: The proof of the first assertion is based on the scaling properties of the measure $d\alpha_m^+(p)$. The next assertions follow from elementary computations. ■

If we use the definition of the annihilation operators

$$\left(a(q;m)\Phi\right)^{(n)}(p_1, \ldots, p_n) = \sqrt{n+1} \quad \Phi^{(n+1)}(q, p_1, \ldots, p_n) \tag{2.0.5}$$

then we immediately get the identity:

$$\mathcal{U}_\lambda \quad a(q;m) \quad \mathcal{U}_\lambda^{-1} = \lambda \quad a(\lambda^{-1}q; \lambda^{-1}m). \tag{2.0.6}$$

By hermitian conjugation we get a similar identity for the creation operators $a^*(q)$.

The expression of the real scalar field of mass m is:

$$\phi(x;m) \equiv \frac{1}{(2\pi)^{3/2}} \int d\alpha_m^+(p) \left[e^{-ix\cdot p} a(p;m) + e^{ix\cdot p} a^*(p;m)\right] \tag{2.0.7}$$

so we get from (2.0.6) the following relation:

$$\mathcal{U}_\lambda \quad \phi(x;m) \quad \mathcal{U}_\lambda^{-1} = \lambda \phi(\lambda x; \lambda^{-1}m). \tag{2.0.8}$$

Remark 2.2 *There is an alternative point of view. One can define the operators* $\mathcal{U}'_\lambda : \mathcal{F}_m \longrightarrow \mathcal{F}_m$ *according to*

$$\left(\mathcal{U}'_\lambda \Phi^{(n)}\right)(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \prod_{i=1}^{n} r_\lambda(\mathbf{p_i}) \quad \Phi^{(n)}(\lambda\mathbf{p}_1, \ldots, \lambda\mathbf{p}_n) \tag{2.0.9}$$

where

$$r_\lambda(\mathbf{p}) \equiv \lambda^{3/2} \sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\lambda\mathbf{p})}}. \tag{2.0.10}$$

Because we have the cocycle identity

$$r_\lambda(\mathbf{p}) \quad r_{\lambda'}(\lambda\mathbf{p}) = r_{\lambda\lambda'}(\mathbf{p}) \tag{2.0.11}$$

the map $\lambda \rightarrow \mathcal{U}'_\lambda$ *defined above is a representation of the multiplicative group* $\mathbb{R}_+$ *(the dilation) group in the Fock space of the scalar field. Moreover, the relations (2.0.4), (2.0.6) and (2.0.8) are valid only up terms of order* $O(m)$ *because we have* $r_\lambda(\mathbf{p}) = \lambda + O(m)$. *So, we see that some information is lost in this approach.*

It is easy to prove that relations of the same type as (2.0.8) are valid for other types of fields, namely fields of integer spin. This includes the electromagnetic potential, the Yang-Mills fields, the gravitational field and also the ghosts fields used in the process of quantization. For a Dirac field an important difference appears. Instead of (2.0.7) we have:

$$\psi(x; M) \equiv \frac{1}{(2\pi)^{3/2}} \int d\alpha^+_M(p) \left[e^{-ix\cdot p} \sum_{i=1}^{2} u_i(p; M) b_i(p; M) + e^{ix\cdot p} \sum_{i=1}^{2} v_i(p; M) b^*_i(p; M) \right] \tag{2.0.12}$$

(see [42]) where $b^{\#}_i(p; M)$ are the creation (annihilation) operators; the expressions $u_i(p; M)$ and $v_i(p; M)$ are solutions of the free Dirac equation of positive (negative) values. To have Poincaré covariance of the field operator ψ one has to normalize in such a way these spinors such that we have:

$$u_i(\lambda p; \lambda M) = \lambda^{1/2} u_i(p; M), \quad v_i(\lambda p; \lambda M) = \lambda^{1/2} v_i(p; M). \tag{2.0.13}$$

So we get instead of (2.0.8):

$$\mathcal{U}_\lambda \quad \psi(x; M) \quad \mathcal{U}_\lambda^{-1} = \lambda^{3/2} \psi(\lambda x; \lambda^{-1} M). \tag{2.0.14}$$

We can obviously prove that the relations (2.0.4) are valid in the most general case, with fields of various spins.

Let us note that if we apply to the relations (2.0.8) or (2.0.14) a derivation operator $\frac{\partial}{\partial x^\mu}$ we obtain a supplementary factor λ in the right hand side.

Finally, if $W(x; \mathbf{m})$ is a Wick monomial in free fields of various masses $\mathbf{m} = (m_1, \ldots, M_1, \ldots)$ we obtain a generalization of the relations (2.0.8) and (2.0.14), namely:

$$\mathcal{U}_\lambda \quad W(x; \mathbf{m}) \quad \mathcal{U}_\lambda^{-1} = \lambda^{\omega(W)} W(\lambda x; \lambda^{-1} \mathbf{m}) \tag{2.0.15}$$

where the number $\omega(W)$ is called the *canonical dimension* of the monomial W and is computed according to the well known rule: one attributes to every integer (resp. half-integer) spin field the canonical dimension 1 (resp. 3/2) and to every derivative the canonical dimension 1. Then one postulates that the canonical dimension is an additive function.

One can extend these considerations to Wick monomials in many variables $W(x_1, \ldots, x_n)$. If the interaction Lagrangian of a model verifies a relation of the type (2.0.15) we say that the model is *dilation* (or *scale*)-*covariant*. It also well known that the canonical dimension of fields is an important property in renormalization theory.

3 Renormalization Theory

3.1 Bogoliubov Axioms

We outline here the axioms of a *multi-Lagrangian* perturbation theory. Following Bogoliubov and Shirkov ideas, in [24] one constructs the S-matrix as a formal series of operator valued distributions:

$$S(\mathbf{g}) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \, T_{j_1,\ldots,j_n}(x_1, \cdots, x_n) g_{j_1}(x_1) \cdots g_{j_n}(x_n), \tag{3.1.1}$$

where $\mathbf{g} = (g_j(x))_{j=1,\ldots P}$ is a multi-valued tempered test function in the Minkowski space $\mathbb{R}^4$ that switches the interaction and $T_{j_1,\ldots,j_n}(x_1, \cdots, x_n)$ are operator-valued distributions acting in the Fock space of some collection of free fields. These operator-valued distributions are called *chronological products* and verify some properties called in the following *Bogoliubov axioms*. It is necessary to note that there is a canonical projection *pr* associating to the point x_i the index j_i. One starts from a set of *interaction Lagrangians* $T_j(x), \quad j = 1, \ldots, P$ and tries to construct the whole series $T_{j_1,\ldots,j_n}, \quad n \geq 2$.

The interaction Lagrangians must satisfy some requirements such like Poincaré invariance, hermiticity and causality. The natural candidates fulfilling these demands is a linearly independent set of Wick polynomials operating in the Fock space (describing a system of weakly interacting particles).

The recursive process of constructing the chronological produces fixes the chronological products almost uniquely. We will study this arbitrariness in detail later.

The physical S-matrix is obtained from $S(\mathbf{g})$ taking the *adiabatic limit* which is , loosely speaking the limit $g_j(x) \rightarrow 1, \quad \forall j = 1, \ldots, P$.

We give here the set of axioms imposed on the chronological products $T_{j_1,\ldots,j_n}$ following the notations of [24].

- First, it is clear that, without loosing generality, we can consider them *completely symmetrical* in all variables in the sense:

$$T_{j_{\pi(1)},\ldots,j_{\pi(n)}}(x_{\pi(1)}, \cdots x_{\pi(p)}) = T_p(x_1, \cdots x_p), \quad \forall \pi \in \mathcal{P}_p. \tag{3.1.2}$$

- Next, we must have *Poincaré invariance*. Because we will also consider Dirac fields, we suppose that we have an unitary representation $(a, A) \mapsto \mathcal{U}_{a,A}$ of the group $inSL(2, \mathbb{C})$ (the universal covering group of the proper orthochronous Poincaré group $\mathcal{P}^{\uparrow}_{+}$) and a finite dimensional representation $A \mapsto S(A)$ of of the group $SL(2, \mathbb{C})$ such that:

$$\begin{gathered}\mathcal{U}_{a,A} T_{j_1,\ldots,j_n}(x_1, \cdots, x_p) \mathcal{U}^{-1}_{a,A} = S(A^{-1})_{j_1 k_1} \cdots S(A^{-1})_{j_n k_n} \times \\ T_{k_1,\ldots,k_n}(\delta(A) \cdot x_1 + a, \cdots, \delta(A) \cdot x_p + a), \quad \forall A \in SL(2, \mathbb{C}), \forall a \in \mathbb{R}^4\end{gathered} \tag{3.1.3}$$

  where $SL(2, \mathbb{C}) \ni A \mapsto \delta(A) \in \mathcal{P}^{\uparrow}_{+}$ is the covering map. In particular, *translation invariance* is essential for implementing Epstein-Glaser scheme of renormalization.

  Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available.

- The central axiom seems to be the requirement of *causality* which can be written compactly as follows. Let us firstly introduce some standard notations. Denote by $V^+ \equiv \{x \in \mathbb{R}^4 | \quad x^2 >$

$0, \quad x_0 > 0\}$ and $V^- \equiv \{x \in \mathbb{R}^4| \quad x^2 > 0, \quad x_0 < 0\}$ the upper (lower) lightcones and by $\overline{V^\pm}$ their closures. If $X \equiv \{x_1, \cdots, x_m\} \in \mathbb{R}^{4m}$ and $Y \equiv \{y_1, \cdots, y_n\} \in \mathbb{R}^{4m}$ are such that $x_i - y_j \notin \overline{V^-}, \quad \forall i = 1, \ldots, m, \quad j = 1, \ldots, n$ we use the notation $X \geq Y$. If $x_i - y_j \notin \overline{V^+} \cup \overline{V^-}, \quad \forall i = 1, \ldots, m, \quad j = 1, \ldots, n$ we use the notations: $X \sim Y$. We use the compact notation $T_J(X) \equiv T_{j_1,\ldots,j_n}(x_1, \cdots, x_n)$ with the convention

$$T_\emptyset(\emptyset) \equiv \mathbf{1} \tag{3.1.4}$$

and by XY we mean the juxtaposition of the elements of X and Y. Then the causality axiom writes as follows:

$$T_{J_1 J_2}(X_1 X_2) = T_{J_1}(X_1) T_{J_2}(X_2), \quad \forall X_1 \geq X_2; \tag{3.1.5}$$

here J_i are the indices corresponding to the the coordinates X_i i.e $J_i \equiv pr(X_i), \quad i = 1, 2$.

¿From (3.1.5) one can derive easily:

$$[T_{J_1}(X_1), T_{J_2}(X_2)] = 0, \quad \text{if} \quad X_1 \sim X_2. \tag{3.1.6}$$

- The *unitarity* of the S-matrix can be expressed if one introduces, the formal series:

$$\bar{S}(\mathbf{g}) = 1 + \sum_{n=1}^\infty \frac{(-i)^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \, \bar{T}_{j_1,\ldots,j_n}(x_1, \cdots, x_n) g_{j_1}(x_1) \cdots g_{j_n}(x_n), \tag{3.1.7}$$

where, by definition:

$$(-1)^{|X|} \bar{T}_J(X) \equiv \sum_{r=1}^{|X|} (-1)^r \sum_{X_1,\ldots,X_r \in Part(X)} T_{J_1}(X_1) \cdots T_{J_r}(X_r); \tag{3.1.8}$$

here $X_1, \cdots, X_r$ is a partition of X, $|X|$ is the cardinal of the set X and the sum runs over all partitions. In the lowest orders we have:

$$\bar{T}_j(x) = T_j(x) \tag{3.1.9}$$

and

$$\bar{T}_{j_1 j_2}(x_1, x_2) = -T_{j_1 j_2}(x_1, x_2) + T_{j_1}(x_1) T_{j_2}(x_2) + T_{j_2}(x_2) T_{j_1}(x_1). \tag{3.1.10}$$

One calls the operator-valued distributions $\bar{T}_{j_1,\ldots,j_n}(x_1, \ldots, x_n)$ *anti-chronological products*. The series (3.1.7) is the inverse of the series (3.1.1) i.e. we have:

$$\bar{S}(\mathbf{g}) = S(\mathbf{g})^{-1} \tag{3.1.11}$$

in the sense of formal series. Then the unitarity axiom is then:

$$\bar{T}_J(X) = T_J(X)^\dagger, \quad \forall X. \tag{3.1.12}$$

One can show that the following relations are identically verified:

$$\sum_{X_1, X_2 \in Part(X)} (-1)^{|X_1|} T_{J_1}(X_1) \bar{T}_{J_2}(X_2) = \sum_{X_1, X_2 \in Part(X)} (-1)^{|X_1|} \bar{T}_{J_1}(X_1) T_{J_2}(X_2) = 0. \tag{3.1.13}$$

Also one has, similarly to (3.1.5):

$$\bar{T}_{J_1 J_2}(X_1 X_2) = \bar{T}_{J_2}(X_2) \bar{T}_{J_1}(X_1), \quad \forall X_1 \geq X_2. \tag{3.1.14}$$

A *renormalization theory* is the possibility to construct such a S-matrix starting from the first order terms: $T_j(x), \quad j = 1,\ldots,P$ which are linearly independent Wick polynomials called *interaction Lagrangians* which should verify the following axioms:

$$\mathcal{U}_{a,A} T_j(x) \mathcal{U}_{a,A}^{-1} = S(A^{-1})_{jk} \quad T_k(\delta(A)\cdot x + a), \quad \forall A \in SL(2,\mathbb{C}), \quad \forall j = 1,\ldots,P \tag{3.1.15}$$

$$[T_j(x), T_k(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad x \sim y, \quad \forall j,k = 1,\ldots,P \tag{3.1.16}$$

and

$$T_j(x)^\dagger = T_j(x), \quad \forall j = 1,\ldots,P. \tag{3.1.17}$$

Usually, these requirements are supplemented by covariance with respect to some discrete symmetries (like spatial and temporal inversions, or PCT), charge conjugations or global invariance with respect to some Lie group of symmetry. Some other restrictions follow from the requirement of the existence of the adiabatic limit, at least in the weak sense.

The case of a *single Lagrangian* perturbation theory corresponds to $P = 1$. In this case the expression $T(x) = T_1(x)$ is the interaction Lagrangian and the chronological products are $T(X) \equiv T_{1\ldots1}(X)$.

More generally, one can consider that the interaction Lagrangian is

$$T(x) = \sum c_j T_j(x) \tag{3.1.18}$$

with c_j some real constants. In this case, the chronological products of the theory are

$$T(X) = \sum c_{j_1} \ldots c_{j_n} T_{j_1,\ldots,j_n}(X). \tag{3.1.19}$$

3.2 Epstein-Glaser Induction

We summarize the steps of the inductive construction of Epstein and Glaser [24], [32]. Let the interaction Lagrangians $T_j(x), j = 1, \ldots, P$ be some linearly independent Wick monomials acting in a certain Fock space with $\omega_j, \quad j = 1, \ldots, P$ the corresponding canonical dimensions. We suppose that they generate the space of all Wick monomials of canonical dimension less that 4. The causality property (3.1.16) is fulfilled, but we must make sure that we also have (3.1.15) and (3.1.17).

Moreover, a certain generalization of the preceding formalism is needed in order express the operator-valued chronological product in terms of numerical distributions [24]. It is convenient let the index j to run from 0 to P and to give, by definition

$$T_0 \equiv \mathbf{1}. \tag{3.2.1}$$

Next, we define the summ $j_1 + j_2$ of two indices $j_1, j_2 = 0, \ldots, P$ through the relation

$$T_{j_1+j_2}(x) =: T_{j_1}(x) T_{j_2}(x) : \tag{3.2.2}$$

and then we extend the summation operation to n-uples of indices $J = (j_1, \ldots, j_n)$ componentwise.

We will use the notation

$$\omega_J \equiv \sum_{j \in J} \omega_j \tag{3.2.3}$$

and we call it the *canonical dimension* of $T_J(X)$.

We suppose that we have constructed the chronological products $T_{j_1,\ldots,j_p}(x_1, \cdots, x_p)$ (for all $p = 1, \ldots, n-1$) having the following properties: (3.1.2), (3.1.5) and (3.1.12) for $p \leq n-1$, (3.1.5) for $|X_1|+|X_2| \leq n-1$ and (3.1.6) for $|X_1|, |X_2| \leq n-1$. Moreover, we suppose that we have the following *Wick expansion* of the chronological products:

$$T_J(X) = \sum_{K+L=J} t_K(X) : T_{l_1}(x_1) \cdots T_{l_n}(x_n) : \tag{3.2.4}$$

for $|X| \leq n-1$; here $t_K(X)$ are numerical distributions (called *renormalized Feynman amplitudes*) with degree of singularity restricted by the following relation:

$$\omega(t_K) \leq \omega_K - 4(n-1). \tag{3.2.5}$$

Let us notice that from (3.2.4) we have:

$$t_J(X) = (\Phi_0, T_J(X)\Phi_0). \tag{3.2.6}$$

We want to construct the distribution-valued operators $T_J(X), \quad |X| = n$ such that the the properties above go from 1 to n. Here are the steps of the construction.

1. One constructs from $T_J(X), \quad |X| \leq n-1$ the expressions $\bar{T}_J(X), \quad |X| \leq n-1$ according to (3.1.8) and proves the properties (3.1.14) for $|X_1| + |X_2| \leq n-1$.

2. Next, we defines the expressions:

$$A'_{j_1,\ldots,j_n}(x_1, \ldots, x_{n-1}; x_n) \equiv {\sum}'_{X_1, X_2 \in Part(X)} (-1)^{|X_2|} T_{J_1}(X_1) \bar{T}_{J_2}(X_2), \tag{3.2.7}$$

$$R'_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n) \equiv {\sum}'_{X_1,X_2\in Part(X)} (-1)^{|X_2|}\bar{T}_{J_1}(X_1)T_{J_2}(X_2) \tag{3.2.8}$$

where the sum $\sum'$ goes over the partitions of $X = \{x_1,\dots,x_n\}$ such that $X_2 \neq \emptyset, \quad x_n \in X_1$. Next, we construct the expression

$$D_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n) \equiv A'_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n) - R'_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n). \tag{3.2.9}$$

and prove that it has causal support i.e. $supp(D_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n)) \subset \Gamma^+(x_n)\cup\Gamma^-(x_n)$ where we use standard notations:

$$\Gamma^{\pm}(x_n) \equiv \{(x_1,\dots,x_n)\in(\mathbb{R}^4)^n | x_i - x_n \in V^{\pm}, \quad \forall i=1,\dots,n-1\}. \tag{3.2.10}$$

3. The distributions $D_J(X)$ can be written in a formula similar to (3.2.4):

$$D_J(X) = \sum_{K+L=J} d_K(X) : T_{l_1}(x_1)\cdots T_{l_n}(x_n) : \tag{3.2.11}$$

where $d_K(X)$ are numerical distributions; in analogy to (3.2.6) we have:

$$d_J(X) = (\Phi_0, D_J(X)\Phi_0)\,. \tag{3.2.12}$$

It follows that the numerical distributions $d_J(X)$ have causal support i.e $supp(d_J(X)) \subset \Gamma^+(x_n)\cup\Gamma^-(x_n)$ and are $SL(2,\mathbb{C})$-invariant. Moreover, their degree of singularity is restricted by

$$\omega(d_K) \leq \omega_K - 4(n-1). \tag{3.2.13}$$

4. There exists a causal splitting

$$d = a - r, \quad supp(a) \subset \Gamma^+(x_n), \quad supp(r) \subset \Gamma^-(x_n) \tag{3.2.14}$$

which is also $SL(2,\mathbb{C})$-invariant and such that the order of the singularity is preserved. So, there exists a $SL(2,\mathbb{C})$-covariant causal splitting:

$$D_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n) = A_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n) - R_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n) \tag{3.2.15}$$

with $supp(A_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n)) \subset \Gamma^+(x_n)$ and $supp(R_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n)) \subset \Gamma^-(x_n)$. The expressions A_n and R_n are the *advanced* (resp. *retarded*) products.

5. We have the relation

$$D_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n)^{\dagger} = (-1)^{n-1} D_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n). \tag{3.2.16}$$

The causal splitting obtained above can be chosen such that

$$A_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n)^{\dagger} = (-1)^{n-1} A_{j_1,\dots,j_n}(x_1,\dots,x_{n-1};x_n). \tag{3.2.17}$$

6. Let us define

$$\begin{aligned} T_{j_1,\ldots,j_n}(x_1,\cdots,x_n) &\equiv A_{j_1,\ldots,j_n}(x_1,\cdots,x_{n-1};x_n) - A'_{j_1,\ldots,j_n}(x_1,\cdots,x_{n-1};x_n) \\ &\equiv R_{j_1,\ldots,j_n}(x_1,\cdots,x_{n-1};x_n) - R'_{j_1,\ldots,j_n}(x_1,\cdots,x_{n-1};x_n). \end{aligned} \tag{3.2.18}$$

Then these expressions satisfy the $SL(2,\mathbb{C})$-covariance, causality and unitarity conditions (3.1.3) (3.1.5) (3.1.6) and (3.1.12) for $p = n$. If we substitute

$$T_{j_1,\ldots,j_n}(x_1,\cdots,x_n) \rightarrow \frac{1}{n!}\sum_{\pi} T_{j_{\pi(1)},\ldots,j_{\pi(n)}}(x_{\pi(1)},\cdots,x_{\pi(n)}) \tag{3.2.19}$$

where the sum runs over all permutations of the numbers $\{1,\ldots,n\}$ then we also have the symmetry axiom (3.1.2). It is easy to see that the induction hypothesis is verified by the operators $T_J(X), \quad |X| = n$ constructed in this way.

3.3 The Arbitrariness of the Chronological Products

This problem was addressed in [5] and [40], as we have already mention in the Introduction. We prefer to give an independent formulation based on the multi-Lagrangian Epstein-Glaser scheme presented above. We consider two solutions of the Bogoliubov axioms with the same "initial conditions" $T_j, \quad j = 1,\ldots,P$ chosen as a basis in the space of Wick monomials of degree 4. We introduce the following notation: if $X = \{x_1,\ldots,x_n\}$ then

$$\delta(X) \equiv \delta(x_1 - x_n)\cdots\delta(x_{n-1} - x_n). \tag{3.3.1}$$

We note the identity:

$$\sum_{x_l \in X} \frac{\partial}{\partial x_l^\mu} \quad \delta(X) = 0. \tag{3.3.2}$$

Now, we have the following result:

Theorem 3.1 *Let $T_J(X)$ and $\tilde{T}_J(X)$ be two solutions of the Bogoliubov axioms such that $T_j(x) = \tilde{T}_j(x), \quad \forall j = 1,\ldots,P$ and both verify the restriction (3.2.5). Then we have the relation:*

$$\tilde{T}_J(X) = T_J(X) + \sum_{r=1}^{|X|-1} \frac{1}{r!} \sum_{X_1,\ldots,X_r \in Part(X)} P_{J_1;k_1}(X_1)\cdots P_{J_r;k_r}(X_r) T_{k_1,\ldots,k_r}(x_{i_1},\ldots,x_{i_r}), \quad \forall |X| \geq 2 \tag{3.3.3}$$

where summation over the indices $k_1,\ldots,k_r = 0,\ldots,P$ is understood, $P_{J;k}(X)$ are distributions of the form

$$P_{J;k}(X) = p_{J;k}(\partial)\delta(X) \tag{3.3.4}$$

with $p_{J;k}(\partial)$ a Lorentz covariant polynomial with constant coefficients in the partial derivatives restricted by:

$$deg(p_{J;k}) + \omega_k \leq \omega_J - 4(n-1) \tag{3.3.5}$$

and $x_{i_p} \in X_p, \forall p = 1,\ldots,r$. In the preceding equation, the convention

$$P_{j;k}(X) \equiv \delta_{jk}, \quad |X| = 1 \tag{3.3.6}$$

is understood.

Proof: We use complete induction. For $n = 2$ one obtains a possible expression $T_{j_1 j_2}(x_1,x_2)$ by causally splitting the distribution $D_{j_1 j_2}(x_1,x_2) = [T_{j_1}(x_1), T_{j_2}(x_2)]$. According to a general result in distribution splitting theory, two such splitting differ by a distribution with support in the set $\{x_1 = x_2\}$ of the type $\sum_{k=0}^{P} [p_{j_1 j_2;k}(\partial)\delta(x_1 - x_2)] T_k(x_2)$; the limitation $deg(p_{j_1 j_2;k}) + \omega_k \leq \omega_{j_1} + \omega_{j_2} - 4$ follows from the restrictions (3.2.5). The Lorentz covariance follows if we make the distribution splitting in a covariant way, which is known to be possible.

We suppose that we have the expressions $P_{J;k}(X), \quad |X| \leq n-1$ such that the formula from the statement is valid for $|X| \leq n-1$; we prove the formula for $|X| = n$. It is convenient to observe that one can write the formula (3.3.3) as follows:

$$\tilde{T}_J(X) = \sum_{r=1}^{|X|} \frac{1}{r!} \sum_{X_1,\ldots,X_r \in Part(X)} P_{J_1;k_1}(X_1)\cdots P_{J_r;k_r}(X_r) T_{k_1,\ldots,k_r}(x_{i_1},\ldots,x_{i_r}), \quad \forall |X| \geq 2 \tag{3.3.7}$$

where the convention (3.3.6) has been used. So, by the induction hypothesis, we have the previous relation for $|X| \leq n-1$.

Let us consider in this case the expression

$$\Delta_J(X) \equiv \tilde{T}_J(X) - \sum_{r=2}^{|X|} \frac{1}{r!} \sum_{X_1,\dots,X_r \in Part(X)} P_{J_1;k_1}(X_1) \cdots P_{J_r;k_r}(X_r) T_{k_1,\dots,k_r}(x_{i_1},\dots,x_{i_r}) \tag{3.3.8}$$

and show that it has the support in the set $x_1 = x_2 = \cdots = x_n$. For this, let us suppose that the point $(x_1,\dots,x_n)$ is outside this set. Then one can find a Cauchy surface separating this set in two non-void subsets Y and Z such that $[Y] \geq [Z]$. Because of the symmetry axiom (3.1.2) we can suppose, without loosing generality, that $Y = \{x_1,\dots,x_i\}$ and $Z = \{x_{i+1},\dots,x_n\}$. In that case, let us notice that in the sum appearing in the preceding formula we can have non-zero contributions only from those partitions $X_1,\dots,X_r$ such that for every $p = 1,\dots,r$ we have either $X_p \subset Y$ or $X_p \subset Z$. This means that for such a choice of $(x_1,\dots,x_n)$ we have:

$$\Delta_J(X) \equiv \tilde{T}_J(X) - \sum_{2 \leq s+t \leq |X|} \sum_{Y_1,\dots,Y_s \in Part(Y)} \frac{1}{s!} \sum_{Z_1,\dots,Z_t \in Part(Z)} \frac{1}{t!}$$
$$P_{J_1;k_1}(Y_1) \cdots P_{J_s;k_s}(Y_s) P_{J_{s+1};k_{s+1}}(Z_1) \cdots P_{J_{s+t};k_{s+t}}(Z_t) T_{k_1,\dots,k_{s+t}}(x_{i_1},\dots,x_{i_{s+t}}) \tag{3.3.9}$$

with $x_{i_p} \in Y_p, \forall p = 1,\dots,s$ and $x_{i_{s+p}} \in Z_p, \forall p = 1,\dots,t$. Now, we can use in the right hand side the causality property (3.1.5) for the chronological products $T_J(X)$ and $\tilde{T}_J(X)$.. We have easily get $\Delta_J(X) = 0$. The support property of the distribution $\Delta_J(X)$ is proved. Using Wick theorem and well known facts about the structure of numerical distribution with support included in the set $x_1 = x_2 = \cdots = x_n$ we get the formula (3.3.3) for $|X| = n$. The Lorentz covariance follows like in the case $n = 2$. This finished the induction. ■

It is clear now why do we need the multi-Lagrangian generalisation of Epstein-Glaser formalism. Even if we work in a theory with a single Lagrangian, the best we can do is to choose it among the set of linearly independent Wick polynomials T_j say, $T(x) = T_1(x)$ and the usual chronological products of a single Lagrangian theory are $T(X) = T_{1\dots1}(X)$ (see the end of Subsection 3.1). To sets of chronological products $T(X)$ and $\tilde{T}(X)$ with the same "initial condition" $T(x) = \tilde{T}(x)$ will be connected by a formula of the following type:

$$\tilde{T}(X) = T(X) + \sum_{r=1}^{|X|-1} \frac{1}{r!} \sum_{X_1,\dots,X_r \in Part(X)} P_{k_1}(X_1) \cdots P_{k_r}(X_r) T_{k_1,\dots,k_r}(x_{i_1},\dots,x_{i_r}), \quad \forall |X| \geq 2 \tag{3.3.10}$$

where we have denoted $P_k(X) \equiv P_{\{1\dots1\};k}(X)$ with $|X|$ entries of the figure 1. So, in the difference between two solutions of the problem will certainly appear other chronological products that $T(X)$.

4 Dilation Covariance of the Chronological Products

4.1 A General Characterization of Dilation Properties

We will use the result from the preceding Subsection to study the generic behaviour of the chronological products with respect to the dilation invariance operators which was defined in Section 2. More explicitly, we consider a certain choice for the chronological products and we will emphasize the mass dependence in an obvious way: $T_J(X;m)$; these expressions are not completely fixed for $|X| \geq 2$ because there is the possibility of finite renormalizations. Nevertheless, we make a concrete choice in accordance with Bogoliubov axioms and we have:

Proposition 4.1 *We suppose that the framework from the preceding Section is valid. Then the following relations are valid for all* $|X| \geq 2$*:*

$$\mathcal{U}_\lambda T_J(X;m)\mathcal{U}_\lambda^{-1} = \lambda^{\omega_J} T_J(\lambda X;\lambda^{-1}m)$$
$$+ \sum_{r=1}^{|X|-1} \frac{1}{r!} \sum_{X_1,\dots,X_r \in Part(X)} P_{J_1;k_1;m;\lambda}(\lambda X_1)\cdots P_{J_r;k_r;m;\lambda}(\lambda X_r) T_{k_1,\dots,k_r}(\lambda x_{i_1},\dots,\lambda x_{i_r};\lambda^{-1}m) \quad (4.1.1)$$

where the distributions $P_{J;k;m;\lambda}(X)$ *are of the form*

$$P_{J;k;m;\lambda}(X) = \sum_\alpha c_{J;k;\alpha}(\lambda,m)\partial^\alpha \delta(X); \quad (4.1.2)$$

here α *are multi-indices and* $|\alpha|$ *is the corresponding length. Moreover, the following relation is verified:*

$$P_{J;k;m;1} = 0 \quad \Longleftrightarrow c_{J;k;\alpha}(1,m) = 0. \quad (4.1.3)$$

Proof: Let us consider the following expressions

$$T_J^\lambda(X) \equiv \lambda^{\omega_J} T_J(X;\lambda^{-1}m), \quad \tilde{T}_J^\lambda(X) \equiv \mathcal{U}_\lambda T_J(\lambda^{-1}X;m)\mathcal{U}_\lambda^{-1}, \quad \forall |X| \geq 2 \quad (4.1.4)$$

both acting in the same Fock space: $\mathcal{F}_{\lambda^{-1}m}$ and having the same "initial conditions"

$$T_j^\lambda(x) \equiv \lambda^{\omega_j} T_j(x), \quad j = 1,\dots,P \quad (4.1.5)$$

due to (2.0.15).

Also, these expressions verify the Bogoliubov axioms: the unitarity and the causality are obvious, but for the Poincaré covariance one had to use the relation (2.0.4). We can apply theorem 3.1 and obtain that the difference between the two expressions $\tilde{T}_J^\lambda(X)$ and $T_J^\lambda(X)$ is a sum of the type appearing in the right hand side of the relation (3.3.3) but with the polynomials depending on the parameter λ. If we make the substitution $X \to \lambda X$ we get the relation from the statement. If we take $\lambda = 1$ then we get (4.1.3). ■

Remark 4.2 *If we change the choice of the chronological products* $T_J(X;m)$, $|X| \geq 2$ *adding some finite renormalizations, then the distributions* $P_{J;k;m;\lambda}(X)$ *will change also by some "coboundary" contribution. We will use this freedom later to simplify their form.*

The central result of this paper describes the explicit λ-dependence of the polynomials appearing in the preceding proposition. We study separately the cases $m \neq 0$ and $m = 0$.

4.2 The Case $m \neq 0$

In this case the following result hold:

Theorem 4.3 *In the case $m \neq 0$ one can choose the chronological products such that the following relations are valid for any $|X| \geq 2$:*

$$\mathcal{U}_\lambda \quad T_J(X;m) \quad \mathcal{U}_\lambda^{-1} = \lambda^{\omega_J} \quad T_J(\lambda X; \lambda^{-1} m). \tag{4.2.1}$$

Proof: Is done by induction.

(i) First, we consider the case $|X| = 2$. We start from the relation (4.1.1) from the preceding proposition and apply $\mathcal{U}_{\lambda'} \cdots \mathcal{U}_{\lambda'}^{-1}$. We easily obtain the cocycle identity

$$P_{J;k;m;\lambda\lambda'}(X) = \lambda^{\omega_J} P_{J;k;\lambda^{-1}m;\lambda'}(X) + (\lambda')^{\omega_k} P_{J;k;m;\lambda}(\lambda'^{-1} X). \tag{4.2.2}$$

If we substitute here the expression (4.1.2) for $P_{J;k;\lambda}(X)$ one finds out immediately from the preceding cocycle identity that we have

$$c(\lambda\lambda'; m) = \lambda^{\omega_J} c(\lambda'; \lambda^{-1} m) + (\lambda')^{4+\omega_k+|\alpha|} c(\lambda; m). \tag{4.2.3}$$

where we omit for simplicity the dependence on J, k and α. More conveniently, one defines

$$d(\lambda, m) \equiv \lambda^{-\omega_J} c(\lambda, m) \tag{4.2.4}$$

and has the cocycle identity:

$$d(\lambda\lambda', m) = d(\lambda', \lambda^{-1} m) + (\lambda')^s d(\lambda, m) \tag{4.2.5}$$

where we have denoted

$$s \equiv 4 + \omega_k + |\alpha| - \omega_J. \tag{4.2.6}$$

¿From (4.1.3) we have the "initial condition":

$$c(1, m) = 0 \quad \Longleftrightarrow \quad d(1, m) = 0. \tag{4.2.7}$$

The equation (4.2.5) can be analysed elementary: if we take $m = \lambda$ the following equation emerges:

$$d(\lambda\lambda', \lambda) = d(\lambda', 1) + (\lambda')^s d(\lambda, \lambda) \tag{4.2.8}$$

or, if we make $\lambda = M$ and $\lambda\lambda' = \Lambda$ we have

$$d(\Lambda, M) = d\left(\frac{\Lambda}{M}, 1\right) + \left(\frac{\Lambda}{M}\right)^s d(M, M). \tag{4.2.9}$$

Now, we substitute this expression into the initial relation (4.2.5) we immediately get

$$d\left(\frac{\lambda}{m}, 1\right) = -\left(\frac{\lambda}{m}\right)^s d(\lambda^{-1} m, \lambda^{-1} m) \tag{4.2.10}$$

which makes sense because $m \neq 0$. Finally, we substitute this expression into (4.2.9) and obtain the most general solution of the cohomological equation (4.2.5)

$$d(\lambda, m) = \lambda^s p(m) - p(\lambda^{-1} m) \tag{4.2.11}$$

where $p(m) = \frac{1}{m^s} d(m,m)$. This proves, in particular, that the general solution of the cohomological equation (4.2.5) is in this case a coboundary. In the end we have

$$c_{J;k;\alpha}(\lambda, m) = \lambda^{4+|\alpha|+\omega_k} p_{J;k;\alpha}(m) - \lambda^{\omega_J} p_{J;k;\alpha}(\lambda^{-1} m) \tag{4.2.12}$$

where $p_{J;k;\alpha}$ are arbitrary functions on m. It follows that the polynomial $P_{J;k;m;\lambda}$ appearing in the relation (4.1.1) for $|X| = 2$ has the generic form

$$P_{J;k;m\lambda}(X) = \sum_\alpha \left[\lambda^{4+|\alpha|+\omega_k} p_{J;k;\alpha}(m) - \lambda^{\omega_J} p_{J;k;\alpha}(\lambda^{-1} m)\right] \partial^\alpha \delta(X). \tag{4.2.13}$$

Now, it is an easy exercise to prove that this expression is a "coboundary" in the sense that if we redefine the chronological products $T_J(X;m)$ in order 2 according to

$$\tilde{T}_J(X, m) \equiv T_J(X, m) - \sum_\alpha p_{J;k;\alpha}(m) \left[\partial^\alpha \delta(X)\right] T_k(x_2, m) \tag{4.2.14}$$

then we will have for $|X| = 2$ a relation of the type (4.1.1) with $P_{J;k;m;\lambda} \longrightarrow \tilde{P}_{J;k;m;\lambda} = 0$. This proves the assertion from the statement in the case $|X| = 2$ that is, we can redefine the chronological products in the second order such that we have the relation from the satement for $|X| = 2$.

(ii) We suppose that the formula from the statement is valid for $2 \leq |X| \leq n-1$ and we prove it for $|X| = n$. The induction hypothesis amounts to

$$P_{J;k;m;\lambda}(X) = 0, \quad |X| \leq n-1. \tag{4.2.15}$$

Then, we get from (4.1.1) for $|X| = n$ the following relation

$$\mathcal{U}_\lambda T_J(X;m) \mathcal{U}_\lambda^{-1} = \lambda^{\omega_J} T_J(\lambda X; \lambda^{-1} m) + P_{J;k;m;\lambda}(\lambda X) T_k(\lambda x_n; \lambda^{-1} m) \quad |X| = n \tag{4.2.16}$$

which is a relation of the same type as the relation for $T_J(X, m), \quad |X| = 2$ obtained above. One can obtain a cohomological relation for the polynomials $P_{J;k;m;\lambda}$ which coincides in fact with (4.2.2). For the coefficients $c_{J;k;\alpha}(\lambda, m)$ we will obtain again an equation of the type (4.2.3):

$$c(\lambda\lambda'; m) = \lambda^{\omega_J} c(\lambda'; \lambda^{-1} m) + (\lambda')^{4(|X|-1)+\omega_k+|\alpha|} c(\lambda; m). \tag{4.2.17}$$

Then we find out that the relation (4.2.5) is valid with (4.2.6) modified to

$$s = 4(|X| - 1) + |\alpha| + \omega_k - \omega_J. \tag{4.2.18}$$

In the end, the most general expression of the polynomial $P_{J;k;m;\lambda}$ is

$$P_{J;k;m\lambda}(X) = \sum_\alpha \left[\lambda^{4(|X|-1)+|\alpha|+\omega_k} p_{J;k;\alpha}(m) - \lambda^{\omega_J} p_{J;k;\alpha}(\lambda^{-1} m)\right] \partial^\alpha \delta(X). \tag{4.2.19}$$

As before, one can make $P_{J;k;\lambda}(X) = 0$ by a suitable redefinition of the chronological products $T_J(X, m), \quad |X| = n$. The induction is finished. ■

4.3 The Case $m = 0$

We remark that in the relation (4.1.1) the distributions $P_{J;k;0;\lambda}$ appear and for simplicity we can skip the entry 0 from the index. We adopt the same convetion for the coefficients $c_{J;k;\alpha}(\lambda, 0)$ appearing in the generic expression (4.1.2).

In this case $m = 0$ we have a much more interesting result:

Theorem 4.4 *In the case $m = 0$ one can redefine the chronological products in such a way that the distributions $P_{J;k;\lambda}(X)$ are of the following form:*

$$P_{J;k;\lambda}(X) = \lambda^{\omega_J} ln(\lambda) \sum_{|\alpha|=\omega_J-4(|X|-1)-\omega_k} c_{J;k;\alpha;\lambda} \partial^\alpha \delta(X). \tag{4.3.1}$$

Proof: As before, is done by induction.

(i) First, we consider the case $|X| = 2$. We start from the relation (4.1.1) from the proposition 4.1 and apply $\mathcal{U}_{\lambda'} \cdots \mathcal{U}_{\lambda'}^{-1}$. We obtain the cocycle identity of the same type as (4.2.2):

$$P_{J;k;\lambda\lambda'}(X) = \lambda^{\omega_J} P_{J;k;\lambda'}(X) + (\lambda')^{\omega_k} P_{J;k;\lambda}(\lambda'^{-1} X) \tag{4.3.2}$$

where now we have no mass dependence. Insetead of the relation (4.2.3) we get:

$$c(\lambda\lambda') = \lambda^{\omega_J} c(\lambda') + (\lambda')^{4+|\alpha|+\omega_k} c(\lambda). \tag{4.3.3}$$

As before, one defines $d(\lambda)$ according to (4.2.4) and has the cocycle identity:

$$d(\lambda\lambda') = d(\lambda') + (\lambda')^s d(\lambda), \quad s \equiv 4 + \omega_k + |\alpha| - \omega_J. \tag{4.3.4}$$

Again we have from (4.1.3) the "initial condition":

$$c(1) = 0 \quad \Longleftrightarrow \quad d(1) = 0. \tag{4.3.5}$$

The equation (4.3.4) can be analysed elementary if we differentiate with respect to λ' and put $\lambda' = 1$. The following differential equation emerges:

$$\lambda d'(\lambda) = d_0 + s d(\lambda) \tag{4.3.6}$$

where $d_0 \equiv d'(1)$. We have two cases:

(a) $\underline{s \neq 0}$

The homogeneous equation $\lambda D'(\lambda) = sD(\lambda)$ has the solution $D(\lambda) = A\lambda^s$. With the methods of variation of constants, we look for a solution of the preceding equation of the form $d(\lambda) = A(\lambda)\lambda^s$ with the initial condition $A(1) = 0$. The function $A(\lambda)$ will verify the equation:

$$A' = d_0 \lambda^{-s-1} \tag{4.3.7}$$

with the solution

$$A(\lambda) = \frac{d_0}{s}(1 - \lambda^{-s}) \tag{4.3.8}$$

¿From here we get the solution

$$d(\lambda) = \frac{d_0}{s}(\lambda^s - 1) \tag{4.3.9}$$

which verifies identically the initial equation (4.3.4).

(b) $s = 0$

The equation (4.3.6) becomes:

$$\lambda d'(\lambda) = d_0 \tag{4.3.10}$$

with the solution $d(\lambda) = d_0 \ln(\lambda)$ which, again, identically verifies the initial equation (4.3.4). We get in this case the solution:

$$c(\lambda) = d_0 \lambda^{\omega_J} ln(\lambda). \tag{4.3.11}$$

In the end, we get, instead of the formula (4.2.13)

$$P_{J;k;m\lambda}(X) = \sum_{|\alpha| \neq \omega_J - 4 - \omega_k} \left[\lambda^{4+|\alpha|+\omega_k} c_{J;k;\alpha} - \lambda^{\omega_J} c_{J;k;\alpha} \right] \partial^\alpha \delta(X)$$
$$+ \sum_{|\alpha| = \omega_J - 4 - \omega_k} \lambda^{\omega_J} ln(\lambda) c_{J;k;\alpha} \partial^\alpha \delta(X). \tag{4.3.12}$$

If we make a redefinition of the chronological products $T_J(X,0), \quad |X| = 2$ we can get rid of the first sum in the preceding expression as in the case $m \neq 0$. This means that one can take

$$P_{J;k;m\lambda}(X) = \lambda^{\omega_J} ln(\lambda) \sum_{\alpha = \omega_J - 4 - \omega_k} c_{J;k;\alpha} \partial^\alpha \delta(X) \tag{4.3.13}$$

which proves the assertion from the statement in the case $|X| = 2$.

(ii) We suppose that the formula from the statement is valid for $2 \leq |X| \leq n-1$ and we prove it for $|X| = n$. We establish a cocycle identity for $P_{J;k;\lambda}(X), \quad |X| = n$. Instead of (4.3.2) we obtain in the same way:

$$P_{J;k;\lambda\lambda'}(X) = \lambda^{\omega_J} P_{J;k;\lambda'}(X) + (\lambda')^{\omega_k} P_{J;k;\lambda}(\lambda'^{-1} X)$$
$$+ \sum_{r=2}^{|X|-1} \frac{1}{r!} \sum_{X_1,\ldots,X_r \in Part(X)} P_{J_1;m_1;\lambda}(\lambda'^{-1} X_1) \cdots P_{J_r;m_r;\lambda}(\lambda'^{-1} X_r) P_{m_1,\ldots,m_r;k;\lambda'}(x_{i_1},\ldots,x_{i_r}) \tag{4.3.14}$$

This relation goes into (4.2.2) for $n = 2$ because the sum disappears. The preceding relation gives, instead of (4.3.3) the following:

$$c(\lambda\lambda') = \lambda^{\omega_J} c(\lambda') + (\lambda')^{4(|X|-1)+\omega_k+|\alpha|} c(\lambda) + (\lambda\lambda')^{\omega_J} ln(\lambda') \sum_{r=2}^{|X|-1} c_r ln^r(\lambda) \tag{4.3.15}$$

where, again, the multi-index α was omitted and c_r are some constants; their value will not be needed. If we define the function $d(\lambda)$ by (4.2.4) we get:

$$d(\lambda\lambda') = d(\lambda') + (\lambda')^s d(\lambda) + ln(\lambda') \sum_{r=2}^{|X|-1} c_r ln^r(\lambda),$$
$$s \equiv 4(|X|-1) + \omega_k + |\alpha| - \omega_J. \tag{4.3.16}$$

We know certainly from the general theorem 4.1 that this equations must have solutions. The only problem is to determine the λ dependence from the preceding equation. As before we get from this relation the differential equation:

$$\lambda d'(\lambda) = d_0 + s d(\lambda) + \sum_{r=2}^{|X|-1} c_r ln^r(\lambda) \tag{4.3.17}$$

We have the same cases as before.

(a) $\underline{s \neq 0}$

The homogeneous equation is again $\lambda D'(\lambda) = sD(\lambda)$ with the the solution $D(\lambda) = A\lambda^s$. If we look for a solution of the equation (4.3.17) of the form $d(\lambda) = A(\lambda)\lambda^s$ with the initial condition $A(1) = 0$ we get for $A(\lambda)$ the equation:

$$A' = \lambda^{-s-1}\left[d_0 + \sum_{r=2}^{|X|-1} c_r ln^r(\lambda)\right]. \tag{4.3.18}$$

In the end, we get, instead of (4.3.9) the following expression for the functions d:

$$d(\lambda) = \frac{d_0}{s}(\lambda^s - 1) + \sum_{r=1}^{|X|-1} a_r ln^r(\lambda) \tag{4.3.19}$$

with a_r some constants. We substitute in the original equation (4.3.16) for the function d and obtain that the only possibility is to have $a_r = 0$ (and $c_r = 0$) so we have in fact the solution:

$$d(\lambda) = \frac{d_0}{s}(\lambda^s - 1). \tag{4.3.20}$$

(b) $\underline{s = 0}$

The equation (4.3.6) becomes:

$$\lambda d'(\lambda) = d_0 + \sum_{r=2}^{|X|-1} c_r ln^r(\lambda). \tag{4.3.21}$$

with the solution

$$d(\lambda) = d_0 \ln(\lambda) + \sum_{r=2}^{|X|-1} \frac{c_r}{r+1} ln^{r+1}(\lambda). \tag{4.3.22}$$

We substitute in the initial equation (4.3.16) and obtain that $c_r = 0$ so

$$d(\lambda) = d_0 \ln(\lambda) \quad \Longleftrightarrow \quad c(\lambda) = d_0 \lambda^{\omega_J} \ln(\lambda). \tag{4.3.23}$$

It follows that the most general expression of the polynomials $P_{J;k;\lambda}(X), \quad |X| = n$ is

$$\begin{aligned} P_{J;k;m\lambda}(X) = & \sum_{|\alpha| \neq \omega_J - 4(|X|-1) - \omega_k} \left[\lambda^{4(|X|-1)+|\alpha|+\omega_k} c_{J;k;\alpha} - \lambda^{\omega_J} c_{J;k;\alpha}\right] \partial^\alpha \delta(X) \\ & + \sum_{|\alpha| = \omega_J - 4(|X|-1) - \omega_k} \lambda^{\omega_J} ln(\lambda) c_{J;k;\alpha} \partial^\alpha \delta(X). \end{aligned} \tag{4.3.24}$$

If we make a redefinition of the chronological products $T_J(X, 0), \quad |X| = n$ we can get rid of the first sum in the preceding expression as in the case $m \neq 0$. This means that one can take

$$P_{J;k;m\lambda}(X) = \lambda^{\omega_J} ln(\lambda) \sum_{|\alpha| = \omega_J - 4(|X|-1) - \omega_k} c_{J;k;\alpha} \partial^\alpha \delta(X) \tag{4.3.25}$$

which proves the assertion from the statement in the case $|X| = n$. The induction is finished. ■

If we substitute the preceding result into the proposition 4.1 we get the following result:

Theorem 4.5 *The following relations are valid for any* $|X| \geq 2$*:*

$$\mathcal{U}_\lambda T_J(X;0)\mathcal{U}_\lambda^{-1} = \lambda^{\omega_J}[T_J(\lambda X;0) + \sum_{r=1}^{|X|-1} \frac{ln^r(\lambda)}{r!} \sum_{X_1,\dots,X_r \in Part(X)} \lambda^{-(\omega_{k_1}+\cdots+\omega_{k_r})} P_{J_1;k_1}(X_1)\cdots P_{J_r;k_r}(X_r) \times T_{k_1,\dots,k_r}(\lambda x_{i_1},\dots,\lambda x_{i_r};0)] \tag{4.3.26}$$

where the distributions $P_{J;k}(X)$ *are of the form*

$$P_{J;k}(X) = \sum_{|\alpha|=\omega_J-4(|X|-1)-\omega_k} c_{J;k;\alpha}\partial^\alpha\delta(X). \tag{4.3.27}$$

We also have:

$$\mathcal{U}_\lambda \bar{T}_J(X;0)\mathcal{U}_\lambda^{-1} = \lambda^{\omega_J}[\bar{T}_J(\lambda X;0) + \sum_{r=1}^{|X|-1} \frac{ln^r(\lambda)}{r!} \sum_{X_1,\dots,X_r \in Part(X)} \lambda^{-(\omega_{k_1}+\cdots+\omega_{k_r})} \bar{P}_{J_1;k_1}(X_1)\cdots \bar{P}_{J_r;k_r}(X_r) \times \bar{T}_{k_1,\dots,k_r}(\lambda x_{i_1},\dots,\lambda x_{i_r};0)]. \tag{4.3.28}$$

Let us also remark that in the case $m = 0$ the operators $\mathcal{U}_\lambda$ act in the same Fock space $\mathcal{F}_0^+$ and so, they form a unitary representation of the dilation group $\mathbb{R}_+$. This means that we can define the infinitesimal generatios of the dilations: let us consider the continuous unitary representation of the additive group $\mathbb{R}$ given by

$$V_\chi \equiv \mathcal{U}_{exp(\chi)} \tag{4.3.29}$$

and denote by D its infinitesimal generator obtained via Stone-von-Neumann theorem:

$$V_\chi = e^{i\chi D}. \tag{4.3.30}$$

Then we have from (2.0.8) the following commutation relation:

$$[D,\phi(x)] \sim -i\left(1 + x^\mu \frac{\partial}{\partial x^\mu}\right)\phi(x). \tag{4.3.31}$$

The infinitesimal form of the relations (2.0.15) and (4.3.26) are:

$$[D,W(x)] \sim -i\left[\omega(W) + x^\mu \frac{\partial}{\partial x^\mu}\right]W(x). \tag{4.3.32}$$

and respectively:

$$[D,T_J(X)] \sim -i\left(\omega_J + \sum_{l\in X} x^\mu \frac{\partial}{\partial x_l^\mu}\right)T_J(X) - i\sum P_{J;k}(X)T_k(x_n). \tag{4.3.33}$$

4.4 Scaling Properties of the Renormalized Feynman Amplitudes

We translate the preceding results for the renormalized Feynman amplitudes. From this analysis one can obtain the asymptotic behaviour of these amplitudes as it is done in the classic paper of Weinberg [47]. We use the expression (3.2.4) for $T_J(X;m)$ emphasizing the mass dependence:

$$T_J(X;m) = \sum_{K+L=J} t_K(X;m) : T_{l_1}(x_1;m) \cdots T_{l_n}(x_n;m) : \tag{4.4.1}$$

and we have:

Theorem 4.6 *The following relations are verified:*
(1) in the case $m \neq 0$:

$$t_J(X;m) = \lambda^{\omega_J} t_J(\lambda X; \lambda^{-1} m); \tag{4.4.2}$$

(2) in the case $m = 0$:

$$t_J(X;0) = \lambda^{\omega_J}[t_J(\lambda X;0) +$$
$$\sum_{r=1}^{|X|-1} \frac{ln^r(\lambda)}{r!} \sum_{X_1,\ldots,X_r \in Part(X)} \lambda^{-(\omega_{k_1}+\cdots+\omega_{k_r})} P_{J_1;k_1}(X_1)\cdots P_{J_r;k_r}(X_r) \times$$
$$t_{k_1,\ldots,k_r}(\lambda x_{i_1},\ldots,\lambda x_{i_r};0)]. \tag{4.4.3}$$

The proof is done using the formula (3.2.6) into the relations (4.3) and (4.3.26). The preceding theorem elucidates the logarithmic behaviour of the renormalized Feynman amplitudes in the case $m = 0$. Presumably, the terms proportional with ln^r correspond to graphs with r loops.

One can obtain the infinitesimal form of the preceding relation: we make $\lambda = e^\chi$, differentiate with respect to the variable χ and put $\chi = 0$. If we take into account that

$$t_j(x) = \delta_{j,0} \tag{4.4.4}$$

the following relations emerges:
(1) for $m \neq 0$:

$$\left(\sum_{l=1}^{n} x_l^\mu \frac{\partial}{\partial x_l^\mu} - m\frac{\partial}{\partial m} + \omega_J\right) t_{J;K}(X;m) = 0. \tag{4.4.5}$$

If we take into account translation invariance, we can express the Feynman amplitudes $t_J(X;m)$ in terms of translation-invariant variables: $\xi_i \equiv x_i - x_n, \quad i = 1,\ldots,n-1$ and we have:

$$\left(\sum_{l=1}^{n-1} \xi_l^\mu \frac{\partial}{\partial \xi_l^\mu} - m\frac{\partial}{\partial m} + \omega_J\right) t_J(\Xi;m) = 0 \tag{4.4.6}$$

or, for the Fourier transform:

$$\left(\sum_{l=1}^{n-1} p_l^\mu \frac{\partial}{\partial p_l^\mu} + m\frac{\partial}{\partial m} - \omega_J\right) \tilde{t}_{J;K}(P;m) = 0. \tag{4.4.7}$$

(2) for $m = 0$:

$$\left(\sum_{l=1}^{n} x_l^\mu \frac{\partial}{\partial x_l^\mu} + \omega_J\right) t_{J;K}(X;m) + P_{J;0}(X) = 0 \tag{4.4.8}$$

or, in translationally invariant variables:

$$\left(\sum_{l=1}^{n-1} \xi_l^\mu \frac{\partial}{\partial \xi_l^\mu} + \omega_J\right) t_J(\Xi;m) + P_{J;0}(\Xi) = 0 \tag{4.4.9}$$

or, for the Fourier transforms:

$$\left(\sum_{l=1}^{n-1} p_l^\mu \frac{\partial}{\partial p_l^\mu} - \omega_J\right) \tilde{t}_{J;K}(P;m) + \tilde{P}_{J;0}(P) = 0. \tag{4.4.10}$$

4.5 The General Case

Suppose that we have a theory with a finite number of fields, some of them of zero-mass and some of non-zero mass. We suppose that there exists at least one field of non-zero-mass. Then one can implement the analysis from the case $m \neq 0$ in the following way. Let us denote the non-zero masses of the theory as $\mathbf{m} \equiv (m_1, \ldots, m_t)$; then we have instead of (4.2.3) the relation:

$$c(\lambda\lambda'; \mathbf{m}) = \lambda^{\omega_J} c(\lambda'; \lambda^{-1}\mathbf{m}) + (\lambda')^{4+\omega_k+|\alpha|} c(\lambda; \mathbf{m}). \tag{4.5.1}$$

It is covenient to work in "polar" coordinates $(m, \mu) \in \mathbb{R}^* \times S^{t-1}$ where $m \equiv |\mathbf{m}|$ and $\mu_i \equiv \frac{m_i}{m}, \quad i = 1, \ldots, t$. Then, the previous relation wites as follows:

$$c(\lambda\lambda'; m, \mu) = \lambda^{\omega_J} c(\lambda'; \lambda^{-1}m, \mu) + (\lambda')^{4+\omega_k+|\alpha|} c(\lambda; m, \mu) \tag{4.5.2}$$

and we see that the variables $\mu \in S^{t-1}$ play no rôle. Then one can implement the proof of the theorem 4.3 without any change. So, it follows that if at least a non-zero mass particle is present in the theory, then the conclusions of the theorem 4.3 and of the case (1) considered above for the numerical distributions are true in this case also. More precisely, we have:

$$\mathcal{U}_\lambda \quad T_J(X; \mathbf{m}) \quad \mathcal{U}_\lambda^{-1} = \lambda^{\omega_J} \quad T_J(\lambda X; \lambda^{-1}\mathbf{m}), \tag{4.5.3}$$

$$\left(\sum_{l=1}^{n} x_l^\mu \frac{\partial}{\partial x_l^\mu} - \sum_{i=1}^{t} m_i \frac{\partial}{\partial m_i} + \omega_J\right) t_{J;K}(X; \mathbf{m}) = 0, \tag{4.5.4}$$

$$\left(\sum_{l=1}^{n-1} \xi_l^\mu \frac{\partial}{\partial \xi_l^\mu} - \sum_{i=1}^{t} m_i \frac{\partial}{\partial m_i} + \omega_J\right) t_J(\Xi; \mathbf{m}) = 0 \tag{4.5.5}$$

and

$$\left(\sum_{l=1}^{n-1} p_l^\mu \frac{\partial}{\partial p_l^\mu} + \sum_{i=1}^{t} m_i \frac{\partial}{\partial m_i} - \omega_J\right) \tilde{t}_{J;K}(P; \mathbf{m}) = 0. \tag{4.5.6}$$

Because these relations follow from scale invariance, they can be called the *Callan-Symanzik equation* in the framework of Epstein-Glaser perturbative scheme. However, we do not obtain the *anomalous dimension* in this way. The usual Callan-Symanzik equation [43], [44] expresses the action of the

(infinitesimal) dilation operator on the generating functional of the Green function of the interacting field, but it is natural to suppose that the two formalisms to be, in some way, equivalent. We will comment more in the last Section about this point where we will indicate the way to connect our result to the standard arguments based on the action principle.

We close this Section with an important remark. The results contained in the Subsections 4.2 and in the general case here are rather natural. Indeed, in the case fron Subsection 4.2 of a fields of mass $m > 0$ one could preceed more directly as follows. One can make a choice for the chronological products $T_J(X; m_0)$ for a certain fixed mass $m_0 > 0$ and **define** the chronological products $T_J(X; m)$ for any other mass m such that one has the simple behaviour described by the equation (4.3). The argument can be immediatedly adapted to the general case of more mass, but with at least one non-zero. It is nevertheless, interesting to work out these cases from pure cohomological considerations. So, it follows that the really non-trivial case is the one when all particles have null masses and when, in principle, one cannot avoid the non-trivial cocycles of logarithmic type.

Let us also remark that another way of proving the result from Subsection 4.2 is by observing that in this case one can apply the central solution for the distribution splitting [42] and in this way the scaling properties of the numerical distributions are preserved.

5 Yang-Mills Theories

In this Section we analyse the scale covariance of the Standard Model (SM) and the consequences of this property for the structure of possible anomalies.

5.1 The Fock Space of the Bosons

We give some basic facts about the quantization of a spin 1 Boson of mass $m > 0$. One can proceed in a rather close analogy to the case of the photon; for more details see [29] and references quoted there. Let us denote the Hilbert space of the Boson by H_m; it carries the unitary representation of the orthochronous Poincaré group $\mathsf{H}^{[m,1]}$.

The Hilbert space of the multi-Boson system should be, as before, the associated symmetric Fock space $\mathcal{F}_m \equiv \mathcal{F}^+(\mathrm{H}_m)$. We construct this Fock space as before in the spirit of algebraic quantum field theory. One considers the Hilbert space $\mathcal{H}^{gh}$ generated by applying on the vacuum Φ_0 the free fields $A^\mu(x), \quad u(x), \quad \tilde{u}(x), \quad \Phi(x)$ which are completely characterize by the following properties:

- Equation of motion:

$$(\square + m^2)A^\mu(x), \quad (\square + m^2)u(x) = 0, \quad (\square + m^2)\tilde{u}(x) = 0, \quad (\square + m^2)\Phi(x) = 0. \tag{5.1.1}$$

- Canonical (anti)commutation relations:

$$\begin{aligned} [A^\mu(x), A^\rho(y)] &= -g^{\mu\rho} D_m(x-y) \times \mathbf{1}, \\ [A^\mu(x), u(y)] = 0, \quad [A^\mu(x), \tilde{u}(y)] &= 0, \quad [A^\mu(x), \Phi(y)] = 0, \\ \{u(x), u(y)\} = 0, \quad \{\tilde{u}(x), \tilde{u}(y)\} = 0, \quad \{u(x), \tilde{u}(y)\} &= D_m(x-y) \times \mathbf{1}, \\ [\Phi(x), \Phi(y)] = D_m(x-y) \times \mathbf{1}, \quad [\Phi(x), u(y)] &= 0. \end{aligned} \tag{5.1.2}$$

- $SL(2,\mathbb{C})$-covariance:

$$\begin{aligned} U_{a,A} A^\mu(x) U_{a,A}^{-1} &= \delta(A^{-1})^\mu{}_\nu A^\nu(\delta(A) \cdot x + a), \\ U_{a,A} u(x) U_{a,A}^{-1} = u(\delta(A) \cdot x + a), \quad U_{a,A} \tilde{u}(x) U_{a,A}^{-1} &= \tilde{u}(\delta(A) \cdot x + a) \\ U_{a,A} \Phi(x) U_{a,A}^{-1} &= \Phi(\delta(A) \cdot x + a) \end{aligned} \tag{5.1.3}$$

- PCT covariance.

$$\begin{aligned} U_{PCT} A_\mu(x) U_{PCT}^{-1} = -A_\mu(-x), \quad U_{PCT} \Phi(x) U_{PCT}^{-1} &= \Phi(-x) \\ U_{PCT} u(x) U_{PCT}^{-1} = -u(-x), \quad U_{PCT} \tilde{u}(x) U_{PCT}^{-1} &= -\tilde{u}(-x), \\ U_{PCT} \Phi_0 &= \Phi_0. \end{aligned} \tag{5.1.4}$$

Remark 5.1 *Although we could give the expressions for $U_{I_s}, \quad U_{I_t}$ and U_C separately, we prefer to give only the expression of the PCT transform because the interaction Lagrangian of the standard model is not invariant with respect to these three operations but it is PCT-covariant.*

We give as before in $\mathcal{H}^{gh}$ the sesqui-linear form $< \cdot, \cdot >$ which is completely characterize by requiring:

$$A_\mu(x)^\dagger = A_\mu(x), \quad u(x)^\dagger = u(x), \quad \tilde{u}(x)^\dagger = -\tilde{u}(x), \quad \Phi(x)^\dagger = \Phi(x). \tag{5.1.5}$$

Now, the expression of the supercharge gets an extra term:

$$Q = \int_{\mathbb{R}^3} d^3x \, [\partial^\mu A_\mu(x) + m\Phi(x)] \overleftrightarrow{\partial_0} u(x) \tag{5.1.6}$$

and one can see that we have

$$[Q, A_\mu] = i\partial_\mu u, \quad \{Q, u\} = 0, \{Q, \tilde{u}\} = -i(\partial_\mu A^\mu + m\Phi), \quad [Q, \Phi] = imu \tag{5.1.7}$$

We still have

$$Q^2 = 0 \Longrightarrow Im(Q) \subset Ker(Q) \tag{5.1.8}$$

and also

$$U_{a,A} Q = Q U_{a,A}, \quad U_{PCT} Q = -Q U_{PCT}. \tag{5.1.9}$$

Finally:

Theorem 5.2 *The sesqui-linear form $< \cdot, \cdot >$ factorizes to a well-defined scalar product on the completion of the factor space $Ker(Q)/Im(Q)$. Then there exists the following Hilbert spaces isomorphism:*

$$\overline{Ker(Q)/Im(Q)} \simeq \mathcal{F}_m; \tag{5.1.10}$$

The representation of the Poincaré group and the PCT operator are factorizing to $Ker(Q)/Im(Q)$ and are producing unitary operators (resp. an anti-unitary operator).

If $\mathcal{W}$ the linear space of all Wick monomials in the fields A_μ, u, $\tilde{u}$ and Φ acting in the Fock space $\mathcal{H}^{gh}$ then the expression of the BRST operator is determined by

$$d_Q u = 0, \quad d_Q \tilde{u} = -i(\partial^\mu A_\mu + m\Phi), \quad d_Q A_\mu = i\partial_\mu u, \quad d_Q \Phi = imu. \tag{5.1.11}$$

and, as a consequence we have

$$d_Q^2 = 0. \tag{5.1.12}$$

If one adds matter fields we proceed as before. In particular, this will mean that the BRST operator acts trivially on the matter fields.

Now we can define the Yang-Mills field. We must consider the case when we have r fields of spin 1 and some of them will have zero mass and the others will be considered of non-zero mass. Apparently, we need the scalar ghosts only in the last case. However it can be shown that with this assumption, there are no non-trivial models. To avoid this situation, we make the following amendment. All the fields considered above will carry an additional index $a = 1, \ldots, r$ i.e. we have the following set of fields: $A_{a\mu}, \ u_a, \ \tilde{u}_a, \ \Phi_a \quad a = 1, \ldots, r$. If one of the fields $A_{a\mu}$ has zero mass we postulate that the corresponding scalar fields Φ_a are physical fields and they will be called *Higgs fields*. Moreover, we do not have to assume that they are massless i.e. if some Boson field A^μ_a has zero mass $m_a = 0$, we can suppose that the corresponding Higgs field Φ_a has a non-zero mass: m^H_a. It is convenient to use the compact notation

$$m^*_a \equiv \begin{cases} m_a & \text{for} \quad m_a \neq 0 \\ m^H_a & \text{for} \quad m_a = 0 \end{cases} \tag{5.1.13}$$

These fields verify the following equations of motion:

$$(\square + m_a^2)A_a^\mu(x) = 0, \quad (\square + m_a^2)u_a(x) = 0, \quad (\square + m_a^2)\tilde{u}_a(x) = 0, \quad (\square + (m_a^*)^2)\Phi_a(x) = 0 \tag{5.1.14}$$

The rest of the formalism stays unchanged. The canonical (anti)commutation relations are:

$$\begin{aligned} [A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x-y) \times \mathbf{1}, \\ \{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1}, \quad [\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m_a^*}(x-y) \times \mathbf{1}; \end{aligned} \tag{5.1.15}$$

and all other (anti)commutators are null. The supercharge is given by

$$Q = \sum_{a=1}^{r} \int_{\mathbb{R}^3} d^3x \, [\partial^\mu A_{a\mu}(x) + m_a \Phi_a(x)] \overleftrightarrow{\partial_0} \, u_a(x) \tag{5.1.16}$$

and verifies all the expected properties.

The Krein operator is determined by:

$$A_{a\mu}(x)^\dagger = A_{a\mu}(x), \quad u_a(x)^\dagger = u_a(x), \quad \tilde{u}_a(x)^\dagger = -\tilde{u}_a(x), \quad \Phi_a(x)^\dagger = \Phi_a(x). \tag{5.1.17}$$

The ghost degree is defined in an obvious way and the expression of the BRST operator is similar to the previous one. In particular we have:

$$d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i(\partial_\mu A_a^\mu + m_a \Phi_a), \quad d_Q A_a^\mu = i\partial^\mu u_a, \quad d_Q \Phi_a = i m_a u_a, \quad \forall a = 1, \ldots, r. \tag{5.1.18}$$

Finally, the condition of gauge invariance is (see [13]):

$$d_Q T(X) = i \sum_{x_l \in X} \frac{\partial}{\partial x_l^\mu} T_l^\mu(X) \tag{5.1.19}$$

for some Wick polynomials $T_l^\mu(X), \quad l = 1, \ldots, |X|$.

5.2 Matter Fields and the Interaction Lagrangian of the SM

In this case the matter field is a set of Dirac fields of mass $M_A, \quad A = 1, \ldots, N$ denoted by $\psi_A(x)$.

These fields are characterized by the following relations [30]; here $A, B = 1, \ldots, N$:

- Equation of motion:
$$(i\gamma \cdot \partial + M_A)\psi_A(x) = 0. \tag{5.2.1}$$

- Canonical (anti)commutation relations:
$$\begin{aligned} [\psi_A(x), A_a^\mu(y)] = 0, \quad [\psi_A(x), u_a(y)] = 0, \quad [\psi_A(x), \tilde{u}_a(y)] = 0, \quad [\psi_A(x), \Phi_a(y)] = 0 \\ \{\psi_A(x), \psi_B(y)\} = 0, \quad \{\psi_A(x), \overline{\psi_B}(y)\} = \delta_{AB} S_{M_A}(x-y) \times \mathbf{1}. \end{aligned} \tag{5.2.2}$$

- Covariance properties with respect to the Poincaré group:
$$U_{a,A} \psi_A(x) U_{a,A}^{-1} = S(A^{-1}) \psi_A(\delta(A) \cdot x + a). \tag{5.2.3}$$

- PCT-covariance:
$$U_{PCT}\psi_A(x)U_{I_s}^{-1} = \gamma_1\gamma_2\gamma_3\overline{\psi_A}(-x)^t. \tag{5.2.4}$$

The condition of gauge invariance remains the same (5.1.19) and one can prove [29] that this condition for $n = 1, 2$ determines quite drastically the interaction Lagrangian of canonical dimension $\omega(T(x)) = 4$:

$$\begin{aligned} T(x) \equiv f_{abc}\,[: A_{a\mu}(x)A_{b\nu}(x)\partial^\nu A_a^\mu(x) : - : A_a^\mu(x)u_b(x)\partial_\mu\tilde{u}_c(x) :]\,, \\ +f'_{abc}\,[: \Phi_a(x)\partial_\mu\Phi_b(x)A_c^\mu(x) : -m_b : \Phi_a(x)A_{b\mu}(x)A_c^\mu(x) : -m_b : \Phi_a(x)\tilde{u}_b(x)u_c(x) :] \\ +f^{"}_{abc} : \Phi_a(x)\Phi_b(x)\Phi_c(x) : +j_a^\mu(x)A_{a\mu}(x) + j_a(x)\Phi_a(x) \end{aligned} \tag{5.2.5}$$

where:
$$j_a^\mu(x) =: \overline{\psi_A}(x)(t_a)_{AB}\gamma^\mu\psi_B(x) : + : \overline{\psi_A}(x)(t'_a)_{AB}\gamma^\mu\gamma_5\psi_B(x) : \tag{5.2.6}$$
and
$$j_a(x) =: \overline{\psi_A}(x)(s_a)_{AB}\psi_B(x) : + : \overline{\psi_A}(x)(s'_a)_{AB}\gamma_5\psi_B(x) : \tag{5.2.7}$$
are the so-called *currents*. The conditions of $SL(2,\mathbb{C})$ and PCT-covariance of the interaction Lagrangian are easy to prove as well as the causality condition. The hermiticity conditions are equivalent to the fact that the complex $N \times N$ matrices $t_a, \quad t'_a, \quad s_a, \quad a = 1,\dots r$ are hermitian and $s'_a, \quad a = 1,\dots,r$ is anti-hermitian. The constants f_{abc} are completely anti-symmetric and verify Jacobi identity so they generate a compact semi-simple Lie group quite naturally. There are other conditions on the rest of the constants as well, but because we do not need these properties in the subsequent analysis, we refer to the literature [29], [30] and references quoted there.

Moreover, it can be proved that the condition of gauge invariance (5.1.19) is valid for $n = 1, 2$ and we can take $T^\mu(x)$ to be of canonical dimension $\omega(T^\mu(x)) = 4$ with the explicit form:

$$\begin{aligned} T^\mu(x) = f_{abc}\left(: u_a(x)A_{b\nu}(x)F_c^{\nu\mu}(x) : -\frac{1}{2} : u_a(x)u_b(x)\partial^\mu(x)\tilde{u}_c(x) :\right) \\ +f'_{abc}\,(m_a : A_a^\mu(x)\Phi_b(x)u_c(x) : + : \Phi_a(x)\partial^\mu\Phi_b(x)u_c(x) :)\,. + u_a(x)j_a^\mu(x). \end{aligned} \tag{5.2.8}$$

The following relations are verified:

- $SL(2,\mathbb{C})$-covariance: for any $A \in SL(2,\mathbb{C})$ we have
$$U_{a,A}T(x)U_{a,A}^{-1} = T(\delta(A)\cdot x + a), \quad U_{a,A}T^\mu(x)U_{a,A}^{-1} = \delta(A^{-1})^\mu{}_\rho T^\rho(\delta(A)\cdot x + a). \tag{5.2.9}$$

- PCT-covariance:
$$U_{PCT}T(x)U_{PCT}^{-1} = T(-x), \quad U_{PCT}T^\mu(x)U_{PCT}^{-1} = T^\mu(-x). \tag{5.2.10}$$

- Causality:
$$[T(x), T(y)] = 0, \quad [T^\mu(x), T^\rho(y)] = 0, \quad [T^\mu(x), T(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad x \sim y. \tag{5.2.11}$$

- Unitarity:
$$T(x)^\dagger = T(x), \quad T^\mu(x)^\dagger = T^\mu(x). \tag{5.2.12}$$

- Ghost content:
$$gh(T(x)) = 0, \quad gh(T^\mu(x)) = 0. \tag{5.2.13}$$

We mention that in [28]-[30], the condition of gauge invariance is analysed up to the order 3.

5.3 Dilation Covariance of the Standard Model

In this Subsection we generalize the arguments from the Sections 2.1 for the standard model. We denote the set of all masses by $\mathbf{m} \equiv (m_a, m_a^*, M_A)_{a=1,\dots,r;A=1,\dots,N}$ and the Fock space of all particles (physical or ghosts) by $\mathcal{H}^{gh}_{\mathbf{m}}$. This Hilbert space is generated from the vacuum by applying the operators: $A^\mu_a(x;m_a), \quad u_a(x;m_a), \quad \tilde{u}_a(x;m_a), \quad \Phi_a(x;m_a^*)$ and $\psi_A(x;M_A)$. We define the dilation operators in the total Hilbert space in analogy to (2.0.2) and the result from the proposition 2.1 stays true; we also have the commutations relations with the Poincaré (2.0.4). Finally, we have from (2.0.8) and (2.0.14):

$$\begin{aligned}
\mathcal{U}_\lambda A^\mu_a(x;m_a)\mathcal{U}_\lambda^{-1} = \lambda A^\mu_a(\lambda x;\lambda^{-1}m_a), \quad \mathcal{U}_\lambda \Phi_a(x;m_a)\mathcal{U}_\lambda^{-1} = \lambda \Phi_a(\lambda x;\lambda^{-1}m_a),\\
\mathcal{U}_\lambda u_a(x;m_a)\mathcal{U}_\lambda^{-1} = \lambda u_a(\lambda x;\lambda^{-1}m_a), \quad \mathcal{U}_\lambda \tilde{u}(x;m_a)\mathcal{U}_\lambda^{-1} = \lambda \tilde{u}(\lambda x;\lambda^{-1}m_a),\\
\mathcal{U}_\lambda \psi_A(x;M_A)\mathcal{U}_\lambda^{-1} = \lambda^{3/2}\psi_A(\lambda x;\lambda^{-1}M_A), \quad \forall a = 1,\dots,r, \quad \forall A = 1,\dots,N.
\end{aligned} \tag{5.3.1}$$

¿From these relations and from the expressions (5.2.5) and (5.2.8) we obtain particular cases of the relation (2.0.15):

$$\mathcal{U}_\lambda T_1(x;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^4 T_1(\lambda x;\lambda^{-1}\mathbf{m}), \quad \mathcal{U}_\lambda T^\mu_1(x;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^4 T^\mu_1(\lambda x;\lambda^{-1}\mathbf{m}); \tag{5.3.2}$$

this means that both expressions have canonical dimension equal to 4 which is also the dimension of the Minkowski space-time. Let us suppose from now on that **there are non-zero masses into the theory**. Then we can apply the argument presented at the end of the previous Section and obtain that for all $|X| \geq 1$ we have the following formulæ:

$$\begin{aligned}
\mathcal{U}_\lambda T(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{4|X|} T(\lambda X;\lambda^{-1}\mathbf{m})\\
\mathcal{U}_\lambda T^\mu_l(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{4|X|} T^\mu_l(\lambda X;\lambda^{-1}\mathbf{m}).
\end{aligned} \tag{5.3.3}$$

We also mention the following result which easily follows from the definitions:

Lemma 5.3 *The following relations is valid for every Wick monomial:*

$$\mathcal{U}_\lambda \left[d_Q W(X;\mathbf{m})\right] \mathcal{U}_\lambda^{-1} = \lambda^{\omega(W)+1} W(\lambda X;\lambda^{-1}\mathbf{m}). \tag{5.3.4}$$

Proof: If the expression W is one of the fields $A^\mu_a(x;m_a), u_a(x;m_a), \tilde{u}_a(x;m_a), \Phi_a(x;m_a^*)$ or $\psi_A(x;M_A), \bar{\psi}_A(x;M_A)$ the formula from the statement follows elementary; then we extend to any Wick monomial by induction, using the derivative properties of the BRST operator. ■

5.4 The Structure of the Anomalies in the Standard Model

We consider the standard model as defined by the Lagrangian (5.2.5). and suppose that there are no anomalies up to the order $n-1$ i.e. we have (5.1.19) up to this order. The purpose of this Subsection is to find if possible anomalous terms can appear in this relation in order n and what limitation are imposed by scale covariance. The analysis is similar to the case of the quantum electrodynamics [32]. However, we prefer to use the formalism developped in Subsections 3.1 and 3.2.

(i) Suppose that we have constructed the chronological products $T_J(X), \quad |X| \leq n-1$ verifying all the induction hypothesis from Subsection 3.2. Then we will be able to use the formulæ of the type (3.2.6).

We must have, in analogy to (3.1.18), a developpment of the type:

$$T^{\mu}(x) = \sum c_j^{\mu} T_j(x) \tag{5.4.1}$$

with c_j^{μ} some real constants; then we will have in analogy to (3.1.19):

$$T_l^{\mu}(X) = \sum c_{j_1} \dots c_{j_l}^{\mu} \dots c_{j_n} T_{j_1,\dots,j_n}(X) \tag{5.4.2}$$

for all $|X| \leq n-1$. In particular, the following conventions hold:

$$T(\emptyset) \equiv \mathbf{1}, \quad T_l^{\mu}(\emptyset) \equiv 0, \quad T_l^{\mu}(X) \equiv 0, \quad \text{for} \quad x_l \notin X. \tag{5.4.3}$$

We supplement the induction hypothesis adding:

- ghost number content:

$$gh(T(X)) = 0, \quad gh(T_l^{\mu}(X)) = 1, \quad |X| \leq n-1; \tag{5.4.4}$$

- gauge invariance:

$$d_Q T(X) = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^{\mu}} T_l^{\mu}(X), \quad |X| \leq n-1. \tag{5.4.5}$$

- scale covariance:

$$\mathcal{U}_{\lambda} T_J(X;\mathbf{m}) \mathcal{U}_{\lambda}^{-1} = \lambda^{\omega_J} T_J(\lambda X; \lambda^{-1}\mathbf{m}), \quad |X| \leq n-1. \tag{5.4.6}$$

(ii) Now we can construct the expressions $D_J(X;\mathbf{m})$ according to the formula (3.2.9) from Subsection 3.2 such that we have the well known properties of causality, Poincaré covariance and unitarity. We consider now a causal splitting of the type (3.2.15) such that we preserve Poincaré covariance and the order of singularity. The chronological products can be obtained from the formula (3.2.18), but we still have some freedom in the choice of the splitting which will shall use in the following. It can be proved as in [32] that we have, instead of the relation (5.4.5) a somewhat weaker form, namely:

$$d_Q A(X;\mathbf{m}) = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^{\mu}} A_l^{\mu}(X;\mathbf{m}) + P(X;\mathbf{m}), \quad |X| = n \tag{5.4.7}$$

where the expressions $A(X;\mathbf{m})$ and $A_l^{\mu}(X;\mathbf{m})$ are constructed from the expressions $A_J(X)$ according to the prescriptions (3.1.19) and (5.4.2). In the right hand side $P(X;\mathbf{m})$ is a Wick polynomial (called *anomaly*) of the following structure:

$$P(X) = \sum_J [p_J(\partial)\delta(X)] : T_{j_1}(x_1) \cdots T_{j_n}(x_n) : \tag{5.4.8}$$

with p_J polynomials in the derivatives with the maximal degree restricted by

$$deg(p_J) + \omega_J \leq 5, \quad \forall J. \tag{5.4.9}$$

If we argue like in theorem 4.3 then we can see that from the induction hypothesis we have the following scaling behaviour of the chronological products $T_J(X), \quad |X| = n$:

$$\mathcal{U}_{\lambda} T_J(X;\mathbf{m}) \mathcal{U}_{\lambda}^{-1} = \lambda^{\omega_J} T_J(\lambda X; \lambda^{-1}\mathbf{m}) + P_{J;k;\mathbf{m};\lambda}(X) T_k(x_n;\mathbf{m}) \tag{5.4.10}$$

for some quasi-local distribution $P_{J;k;\mathbf{m};\lambda}(X)$ having an expression of the form (3.3.4). Moreover, these distributions have a coboundary structure (because we are in the case when we have massive fields in the theory) so one can redefine the chronological products $T_J(X)$ such that one gets rid of the expression $P_{J;k;\mathbf{m};\lambda}(X)$. This implies, redefinitions for the causal splitting (3.2.15), in particular redefinitions for the distributions $A(X;\mathbf{m})$ and $A^\mu_l(X;\mathbf{m})$. It is clear that in this way we will not affect the general structure of the equation (5.4.7), that is we eventually modify the anomaly P without spoiling the Poincaré covariance and the order of singularity of the splitting. Moreover, we will have in this way, instead of (5.4.10), the relation (5.4.6) for $|X| = n$ also. Because we can prove from the induction hypothesis that we have

$$\mathcal{U}_\lambda A'_J(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{\omega_J} A'_J(\lambda X;\lambda^{-1}\mathbf{m}) \tag{5.4.11}$$

for $|X| = n$, we obtain that in this case we have also

$$\mathcal{U}_\lambda A_J(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{\omega_J} A_J(\lambda X;\lambda^{-1}\mathbf{m}) \tag{5.4.12}$$

and in particular:

$$\mathcal{U}_\lambda A(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{4n} A(\lambda X;\lambda^{-1}\mathbf{m}), \quad \mathcal{U}_\lambda A^\mu_l(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{4n} A^\mu_l(\lambda X;\lambda^{-1}\mathbf{m}). \tag{5.4.13}$$

This result can be obtained combining with the gauge invariance condition given by the equation (5.4.7) in the following way: we apply to $\mathcal{U}_\lambda \cdots \mathcal{U}_\lambda^{-1}$ to (5.4.7), we use the lemma 5.3.4 and then the previous relations. The result is the following identity verified by the anomaly:

$$\mathcal{U}_\lambda P(X;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^{4n+1} P(\lambda X;\lambda^{-1}\mathbf{m}). \tag{5.4.14}$$

In other words, the anomaly must have the canonical dimension equal to $4n+1$. By "integrations by parts" (see [32]) we can exhibit the anomaly as follows:

$$P(X) = i \sum_{l=1}^{n} \frac{\partial}{\partial x^\rho_l} N^\rho_l(X) + P'(X) \tag{5.4.15}$$

where $P'(X)$ is of the following form:

$$P'(X) = \delta(X)\mathcal{P}(x_n) \tag{5.4.16}$$

with $\mathcal{P}(x)$ a Wick polynomial in one variable. So, by redefining the expressions $A^\mu_l(X)$ we can take the anomaly of the form

$$P(X) = \delta(X)\mathcal{P}(x_n). \tag{5.4.17}$$

It is obvious that the "integration by parts" process will not affect the properties of the anomaly that we have already obtained. In consequence, the Wick polynomial $\mathcal{P}(x)$ will verify the following restrictions:

- $SL(2,\mathbb{C})$-covariance:

$$U_{a,A}\mathcal{P}(x)U_{a,A}^{-1} = \mathcal{P}(\delta(A)\cdot x + a), \quad \forall (a,A) \in inSL(2,\mathbb{C}). \tag{5.4.18}$$

- Ghost numbers restrictions:

$$gh(\mathcal{P}(x)) = 1. \tag{5.4.19}$$

- Scale covariance:

$$\mathcal{U}_\lambda \mathcal{P}(x;\mathbf{m})\mathcal{U}_\lambda^{-1} = \lambda^5 \mathcal{P}(\lambda X; \lambda^{-1}\mathbf{m}). \tag{5.4.20}$$

If we use the generic structure

$$\mathcal{P}(x;\mathbf{m}) = \sum_j c_j(\mathbf{m}) \quad T_j(x;\mathbf{m}) \tag{5.4.21}$$

with $c_j(\mathbf{m})$ some mass-dependent constants, in the last equation we immediately get:

$$c_j(\lambda\mathbf{m}) = \lambda^{5-\omega_j} c_j(\mathbf{m}). \tag{5.4.22}$$

If we now construct the chronological products $T(X), \quad T_l^\mu(X)$ one can fix in a standard way (see [24]) the properties of symmetrization and unitarity without spoiling the other relations we have already obtained. In the same one can fix PCT-covariance. In the end, we will have a relation of the type:

$$d_Q T(X;\mathbf{m}) = i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T_l^\mu(X;\mathbf{m}) + P(X;\mathbf{m}), \quad |X| = n \tag{5.4.23}$$

where the anomaly $P(X,\mathbf{m})$ has the form (5.4.17) and verifies all the preceding restrictions: Poincaré covariance, ghost number restriction, scale covariance. Moreover, it obviously verifies gauge invariance:

$$d_Q P(X;\mathbf{m}) = i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} P_l^\mu(X;\mathbf{m}) \tag{5.4.24}$$

for some Wick polynomials $P_l^\mu(X;\mathbf{m})$ and also PCT-covariance:

$$U_{PCT}\mathcal{P}(x)U_{PCT}^{-1} = (-1)^n \mathcal{P}(-x) \tag{5.4.25}$$

and unitarity:

$$\mathcal{P}(x)^\dagger \equiv (-1)^n \mathcal{P}(x). \tag{5.4.26}$$

(iii) The list of possible anomalies can be written now as in [32]. We only remark that the restrictions imposed above do not lead to the conclusion that there are no anomalies in order n. In fact, a number of *hard anomalies* remain such as:

$$\mathcal{P}_1 = c^1_{abcde} \sum_{m_a=m_b=m_c=m_d=m_e=0} u_a : \Phi_b \Phi_c \Phi_d \Phi_e : \tag{5.4.27}$$

$$\mathcal{P}_2 = c^2_{abc} \sum_{m_a=m_b=m_c=0} u_a : \partial^\mu \Phi_b \partial_\mu \Phi_c : \tag{5.4.28}$$

$$\mathcal{P}_3 = c^3_{abc} \varepsilon_{\mu\nu\rho\sigma} u_a : F_b^{\mu\nu} F_c^{\sigma\rho} : \tag{5.4.29}$$

where

$$F_a^{\mu\nu} \equiv \partial^\mu A_a^\nu - \partial^\nu A_a^\mu. \tag{5.4.30}$$

$$\mathcal{P}_4 = \sum_{m_a=m_b=0} \left[: \bar\psi_A (K_{ab})_{AB} \psi_B :: \bar\psi_A (K'_{ab})_{AB} \gamma_5 \psi_B :\right] u_a \Phi_b. \tag{5.4.31}$$

One can show that from unitarity (or PCT-covariance) that we have

$$c_{\dots} = (-1)^n c_{\dots}, \quad K^*_{ab} = (-1)^n K_{ab}, \quad (K'_{ab})^* = (-1)^n K'_{ab}. \tag{5.4.32}$$

The list of hard anomalies is larger: all the anomalies appearing in the second and in the third order of perturbation theory (see [29] and [30]) should appear.

6 Conclusions

The expression (5.4.29) is the famous Adler-Bardeen-Bell-Jackiw anomaly (ABBJ). So, we see that the various symmetries of the standard model (including scale covariance) are not sufficient to prove the anomalies are absent in higher orders of the perturbation theory if they are absent in orders $n = 1, 2, 3$ (at least in Epstein-Glaser approach). In fact, if a certain type of anomaly is present in low orders of perturbation theory, this means that the corresponding expression is not in conflict with the various symmetries of the model. Then it is hard to imagine why such a conflict would appear in higher orders of perturbation theory. Such a result would be possible in our formalism only if in the equation (5.4.22) the number n (the order of the perturbation theory) would survive.

To obtain the cancelation of anomalies in all orders in our formalism a more refined formula for the distribution splitting seems to be needed.

There appears to be a contradiction between our result and the analysis from [3] (see also [4] and [41]) where it is showed that the ABBJ anomaly can appear only in the order $n = 3$.

The discrepancy can be explained if one admits that in these references one works with the **interaction fields**. One know that one can construct such fields from the chronological products $T_J(X)$ as formal series (see formula (76) of [24]) of the type

$$\Phi(x) = \sum \frac{i^n}{n!} \int dy_1 \dots dy_n R(y_1, \dots, y_n; x) g(y_1) \cdots g(y_n) \tag{6.0.33}$$

where R are the retarded products. If we perform formally the adiabatic limit we have

$$\Phi(x) = \sum_{n=0}^{\infty} g^n \ \phi_n(x) \tag{6.0.34}$$

where ϕ_0 is the free field and g is the coupling constant.

Now supposes that a formula of the following type is valid:

$$\mathcal{U}_\lambda \quad \Phi(x) \quad \mathcal{U}_\lambda^{-1} = \lambda^{\omega-\gamma(g)} \quad \Phi(\lambda x), \tag{6.0.35}$$

where ω is the canonical dimension of the field ϕ and

$$\gamma(g) = \sum_{n=1}^{\infty} \gamma_n g^n \tag{6.0.36}$$

is a formal series in the coupling constant called *anomalous dimension.* One can write

$$\lambda^{-\gamma(g)} = e^{-\gamma(g) ln(\lambda)} \tag{6.0.37}$$

perform the Taylor expansion in g and substitute in the preceding formula. Then one finds out by regrouping the terms that we have

$$\mathcal{U}_\lambda \quad \phi_n(x) \quad \mathcal{U}_\lambda^{-1} = \lambda^\omega \quad \sum_{p+q=n} \sum_{m_1,\ldots,m_p} \frac{(-ln\lambda)^{m_1+\cdots m_p}}{p!} \gamma_{m_1} \cdots \gamma_{m_p} \Phi_q(\lambda x). \tag{6.0.38}$$

This relation should be interpreted as a relation on the retarded products R. In fact such a relation is compatible with our analysis in the scaling limit, where all momenta are very large. In this region it is plausible to assume that all masses of the theory are zero, so we can apply the result of Subsection 4.3 which leads to a formula having the structure (6.0.38).

It is an interesting problem to make this analysis completely rigourous in the framework of Epstein-Glaser formalism.

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References

[1] A. Aste, G. Scharf, "*Non-Abelian Gauge Theories as a Consequence of Perturbative Quantum Gauge Invariance*", hep-th/9803011, ZU-TH-6/98

[2] A. Aste, G. Scharf, M. Dütsch, "*Perturbative Gauge Invariance: the Electroweak Theory II*, Ann. Phys. (Leipzig) **8** (1999) 389-404, hep-th/9702053

[3] G. Bandelloni, C. Becchi, A. Blasi, R. Collina, "*On the Cancellation of hard Anomalies in Gauge Field Models: a Regularization Independent Proof*", Commun. Math. Phys. **72** (1980) 239-272

[4] C. Becchi, A. Rouet, R. Stora, "*Renormalizable Theories with Symmetry Breaking*", in "Field Theory, Quantization and Statistical Physics", ed. E. Tirapegui, MPAM series vol. 6, D. Reidel Publ. Comp. pp. 3-32

[5] N. N. Bogoliubov, D. Shirkov, "*Introduction to the Theory of Quantized Fields*", John Wiley and Sons, 1976 (3rd edition)

[6] C. G. Callan jr., "*Broken Scale Invariance in Scalar Field Theory*", Phys. Rev. **D 2** (1970) 1541-1547

[7] C. G. Callan jr., S. Coleman, R. Jackiw, "*A New Improved Energy-Momentum Tensor*", Ann. of Phys. (N.Y.) **59** (1970) 42-73

[8] S. Coleman, R. Jackiw, "*Why Dilation Generators Do Not generate Dilatations*", Ann. of Phys. (N.Y.) **67** (1971) 552-598

[9] G. Costa, T. Marinucci, M. Tonin, J. Julve, "*Non-Abelian Gauge Theories and Triangle Anomalies*", Il Nuovo Cimento **38 A** (1977) 373-412

[10] M. Dütsch, "*On Gauge Invariance of Yang-Mills Theories with Matter Fields*", Zürich-University-Preprint ZU-TH-10/95, Il Nuovo Cimento **A 109** (1996) 1145-1186

[11] M. Dütsch, "*Slavnov-Taylor Identities from the Causal Point of View*", Zürich-University-Preprint ZU-TH-30/95, hep-th/9606105, Int. Journ. Mod. Phys. **A 12** (1997) 3205-3248

[12] M. Dütsch, T. Hurth, F. Krahe, G. Scharf, "*Causal Construction of Yang-Mills Theories. I*", Il Nuovo Cimento **A 106** (1993) 1029-1041

[13] M. Dütsch, T. Hurth, F. Krahe, G. Scharf, "*Causal Construction of Yang-Mills Theories. II*", Il Nuovo Cimento **A 107** (1994) 375-406

[14] M. Dütsch, T. Hurth, G. Scharf, "*Gauge Invariance of Massless QED*", Phys. Lett **B 327** (1994) 166-170

[15] M. Dütsch, T. Hurth, G. Scharf, "*Causal Construction of Yang-Mills Theories. III*", Il Nuovo Cimento **A 108** (1995) 679-707

[16] M. Dütsch, T. Hurth, G. Scharf, "*Causal Construction of Yang-Mills Theories. IV. Unitarity*", Il Nuovo Cimento **A 108** (1995) 737-773

[17] M. Dütsch, F. Krahe G. Scharf, "*Interacting Fields in Finite QED*", Il Nuovo Cimento, **A 103** (1990) 871-901

[18] M. Dütsch, F. Krahe G. Scharf, "*Gauge Invariance in Finite QED*", Il Nuovo Cimento, **A 103** (1990) 903-925

[19] M. Dütsch, F. Krahe G. Scharf, "*Axial Anomalies in Massless Finite QED*", Il Nuovo Cimento, **A 105** (1992) 399-422

[20] M. Dütsch, F. Krahe G. Scharf, "*The Vertex Function and Adiabatic Limit in Finite QED*", J. Phys. **G 19** (1993) 485-502

[21] M. Dütsch, F. Krahe G. Scharf, "*The Infrared Problem and Adiabatic Switching*", J. Phys. **G 19** (1993) 503-515

[22] M. Dütsch, F. Krahe G. Scharf, "*Scalar QED Revisited*", Il Nuovo Cimento **A 106** (1993) 277-307

[23] M. Dütsch, G. Scharf "*Perturbative Gauge Invariance: the Electroweak Theory*, Ann. Phys. (Leipzig) **8** (1999) 359-387, hep-th/9612091

[24] H. Epstein, V. Glaser, "*The Rôle of Locality in Perturbation Theory*", Ann. Inst. H. Poincaré **19 A** (1973) 211-295

[25] V. Glaser, "*Electrodynamique Quantique*", L'enseignement du 3e cycle de la physique en Suisse Romande (CICP), Semestre d'hiver 1972/73

[26] N. Grillo, "*Some Aspects of Quantum Gravity in the Causal Approach*", hep-th/9903011, the 4th workshop on "Quantum Field Theory under the Influence of External Conditions", Leipzig, Germany, Sept. 1998

[27] N. Grillo, "*Causal Quantum Gravity*", hep-th/9910060

[28] D. R. Grigore, "*On the Uniqueness of the Non-Abelian Gauge Theories in the Epstein-Glaser Approach to Renormalization Theory*", hep-th/9806244, submitted for publication

[29] D. R. Grigore, "*The Standard Model and its Generalizations in the Epstein-Glaser Approach to Renormalization Theory*", hep-th/9810078, submitted for publication

[30] D. R. Grigore "*The Standard Model and its Generalisations in Epstein-Glaser Approach to Renormalization Theory II: the Fermion Sector and the Axial Anomaly* ", hep-th/9903206,

[31] D. R. Grigore, "*On the Quantization of the Gravitational Field*", hep-th/9905190, Class. Quant. Gravity **17** (2000) 319-344

[32] D. R. Grigore, "*Gauge Invariance of the Quantum Electrodynamics in the Causal Approach to Renormalization Theory*", hep-th/9911214

[33] T. Hurth, "*Nonabelian Gauge Theories. The Causal Approach*", Zürich-University-Preprint ZU-TH-36/94, hep-th/9511080, Ann. of Phys. **244** (1995) 340-425

[34] T. Hurth, "*Nonabelian Gauge Symmetry in the Causal Epstein-Glaser Approach*", Zürich-University-Preprint ZU-TH-20/95, hep-th/9511139, Int. J. Mod. Phys. **A 12** (1997) 4461-4476

[35] T. Hurth, "*A Note on Slavnov-Taylor Identities in the Causal Epstein-Glaser Approach*", Zürich-University-Preprint ZU-TH-21/95, hep-th/9511176

[36] T. Hurth, "*Higgs-free Massive Nonabelian Gauge Theories*", Zürich-University-Preprint ZU-TH-25/95, hep-th/9511176, Helv. Phys. Acta **70** (1997) 406-416

[37] F. Krahe, "*A Causal Approach to Massive Yang-Mills Theories*", DIAS-STP-95-01

[38] F. Krahe, "*Causal Perturbation Theory for Massive Vector Boson Theories*", Acta Phys. Polonica **B 27** (1996) 2453-2476

[39] J. H. Lowenstein, "*Differential Vertex Operations in Lagrangian Field Theory*", Commun. Math. Phys. **24** (1971) 1-21

[40] G.Pinter, "*The Action Principle in Epstein-Glaser Renormalization and Renormalization of the S-Matrix of* Φ^4*-Theory*", hep-th/9911063

[41] O. Piguet, A. Rouet, "*Symmetries in Perturbative Quantum Field Theory*", Phys. Rep. **76** (1981) 1-77

[42] G. Scharf, "*Finite Quantum Electrodynamics: The Causal Approach*", (second edition) Springer, 1995

[43] K. Symanzik, "*Small-Distance Behaviour in Field Theory and Power Counting*", Commun. Math. Phys. **18** (1970) 227-246

[44] K. Symanzik, "*Small-Distance Behaviour Analysis and Wilson Expansions*", Commun. Math. Phys. **23** (1971) 49-86

[45] G. Scharf, M. Wellmann, "*Quantum Gravity from Perturbative Gauge Invariance*", hep-th/9903055

[46] V. S. Varadarajan, "*Geometry of Quantum Theory*", (second edition) Springer, 1985

[47] S. Weinberg, "*High-Energy Behavior in Quantum Field Theory*", Phys. Rev. **118** (1960) 838-849